1. Find all points on the polar curve $r = 1 + \cos\theta$ where the tangeng line is (a) horizontal and (b) vertical.

   Solution:

$$\begin{aligned}\frac{dx}{d\theta} &= -(1+\cos\theta)\sin\theta - \sin\theta\cos\theta \\ &= -\sin\theta - 2\sin\theta\cos\theta \\ &= -\sin\theta\,(1+2\cos\theta)\,, \\ \frac{dy}{d\theta} &= (1+\cos\theta)\cos\theta - \sin^2\theta \\ &= 2\cos^2\theta + \cos\theta - 1 \\ &= (2\cos\theta - 1)(\cos\theta + 1)\end{aligned}$$

   For horizontal tangent,

$$\begin{aligned}\frac{dy}{d\theta} &= 0,\ \frac{dx}{d\theta} \neq 0 \\ \cos\theta &= \frac{1}{2},\ \cos\theta = -1 \\ \theta &= \frac{\pi}{3}, \frac{5\pi}{3}, \pi\end{aligned}$$

   The third anser is rejected since it makes $\frac{dx}{d\theta} = 0$. Points of horizontal tangents are $\left(\frac{1}{2}, \frac{\pi}{3}\right), \left(\frac{1}{2}, \frac{5\pi}{3}\right)$.

   For vertical tangents

$$\begin{aligned}\frac{dx}{d\theta} &= 0,\ \frac{dy}{d\theta} \neq 0 \\ \sin\theta &= 0, \cos\theta = -\frac{1}{2} \\ \theta &= 0, \pi, \frac{2\pi}{3}, \frac{4\pi}{3}.\end{aligned}$$

   The second answer is rejected since it makes $\frac{dy}{d\theta} = 0$. Points of vertical tangents are $(2,0), \left(\frac{1}{2}, \frac{2\pi}{3}\right), \left(\frac{1}{2}, \frac{4\pi}{3}\right)$.

2. Find the length of the polar curve $r = e^{2\theta},\ 0 \leq \theta \leq 2\pi$

Solution:

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(e^{2\theta}\right)^2 + \left(2e^{2\theta}\right)^2} d\theta \\ &= \int_0^{2\pi} \sqrt{e^{4\theta} + 4e^{4\theta}} d\theta \\ &= \int_0^{2\pi} \sqrt{5e^{4\theta}} d\theta \\ &= \sqrt{5} \int_0^{2\pi} e^{2\theta} d\theta \\ &= \frac{\sqrt{5}}{2} \left(e^{2\theta}\right)_0^{2\pi} \\ &= \frac{\sqrt{5}}{2} \left(e^{4\pi} - 1\right). \end{aligned}$$

3. Find x such that the points $P(x, 0, 1)$, $Q(2, 4, 6)$, $R(3, -1, 2)$ and $S(6, 2, 8)$ lie in the same plane.

   Solution:

$$\overrightarrow{PQ} = \langle 2 - x, 4, 5 \rangle, \ \overrightarrow{PR} = \langle 3 - x, -1, 1 \rangle, \ \overrightarrow{PS} = \langle 6 - x, 2, 7 \rangle.$$

   For $\overrightarrow{PQ}$, $\overrightarrow{PR}$, $\overrightarrow{PS}$ to be coplanar, we must have $\overrightarrow{PQ} \cdot \left(\overrightarrow{PR} \times \overrightarrow{PS}\right) = 0$ or

$$\begin{aligned} \begin{vmatrix} 2 - x & 4 & 5 \\ 3 - x & -1 & 1 \\ 6 - x & 2 & 7 \end{vmatrix} &= 0 \\ 18x - 18 &= 0 \\ x &= 1. \end{aligned}$$

4. (a) Find the center and radius of the sphere $x^2 + y^2 + z^2 = 4x - 2y$.

   Solution:

$$\begin{aligned} \left(x^2 - 4x + 4\right) + \left(y^2 + 2y + 1\right) + z^2 &= 4 + 1 \\ (x - 2)^2 + (y + 1)^2 + z^2 &= 5. \end{aligned}$$

   Center $(2, -1, 0)$, radius $\sqrt{5}$.

   (b) Find an equation of the largest sphere with center at $(6, 2, 3)$ that is contained in the first octant.

   Solution:

   The center is closest to the xz-plane. The largest sphere must have radius 2. The equation of that sphere is

$$(x - 6)^2 + (y - 2)^2 + (z - 3)^2 = 4.$$

5. (a) Find $|\mathbf{a}|$, $\mathbf{a}+\mathbf{b}$, $\mathbf{a}-\mathbf{b}$ and $3\mathbf{a}+4\mathbf{b}$ for $\mathbf{a}=\langle -3,-4,-1\rangle$, $\mathbf{b}=\langle -1,5,-2\rangle$.

Solution:

$$\begin{aligned} |\mathbf{a}| &= |\langle -3,-4,-1\rangle| = \sqrt{9+16+1} = \sqrt{26}. \\ \mathbf{a}+\mathbf{b} &= \langle -3,-4,-1\rangle + \langle -1,5,-2\rangle = \langle -4,1,-3\rangle \\ \mathbf{a}-\mathbf{b} &= \langle -3,-4,-1\rangle - \langle -1,5,-2\rangle = \langle -2,-9,1\rangle \\ 3\mathbf{a}+4\mathbf{b} &= 3\langle -3,-4,-1\rangle + 4\langle -1,5,-2\rangle = \langle -13,8,-11\rangle . \end{aligned}$$

(b) Find a vector that has the same direction as $\langle -2,4,2\rangle$ but has lenght 6

Solution:

$$\begin{aligned} \mathbf{v} &= 6\frac{\langle -2,4,2\rangle}{|\langle -2,4,2\rangle|} = \frac{6}{\sqrt{24}}\langle -2,4,2\rangle \\ &= \sqrt{6}\langle -1,2,1\rangle . \end{aligned}$$

6. Find the scalar and vector projections of the vector $\mathbf{v}=2\mathbf{i}-3\mathbf{j}+\mathbf{k}$ onto the vector $\mathbf{w}=\mathbf{i}-2\mathbf{j}$.

Solution:

$$\begin{aligned} \mathrm{Comp}_{\mathbf{w}}\mathbf{v} &= \frac{\mathbf{w}\cdot\mathbf{v}}{|\mathbf{w}|} = \frac{8}{\sqrt{5}}, \\ \mathrm{Pr\,oj}_{\mathbf{w}}\mathbf{v} &= \mathrm{Comp}_{\mathbf{w}}\mathbf{v}\frac{\mathbf{w}}{|\mathbf{w}|} = \frac{8}{5}(\mathbf{i}-2\mathbf{j}) \\ &= \left(\frac{8}{5}\mathbf{i}-\frac{16}{5}2\mathbf{j}\right) \end{aligned}$$