1. (a) Is the function

$$f(x,y)=\begin{cases}\frac{xy}{x^2+xy+y^2} & (x,y)\neq(0,0)\\ \frac{1}{3} & (x,y)=(0,0)\end{cases}$$

continuous at $(0,0)$? Why?

Solution:

Approach $(0,0)$ through $x=0$ to get $\lim_{y\to0} f(0,y)=0.$

Approach $(0,0)$ through $x=y$ to get $\lim_{y\to0} f(y,y)=\frac{1}{3}.$

Since $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exit, f is not continuous at $(0,0)$.

(b) Calculate $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$.

Solution:

Using polar coordinates, we put $x=r\cos\theta, y=r\sin\theta$. This gives $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}=\lim_{r\to0} r\left(\cos^3\theta+\sin^3\theta\right)$. Since

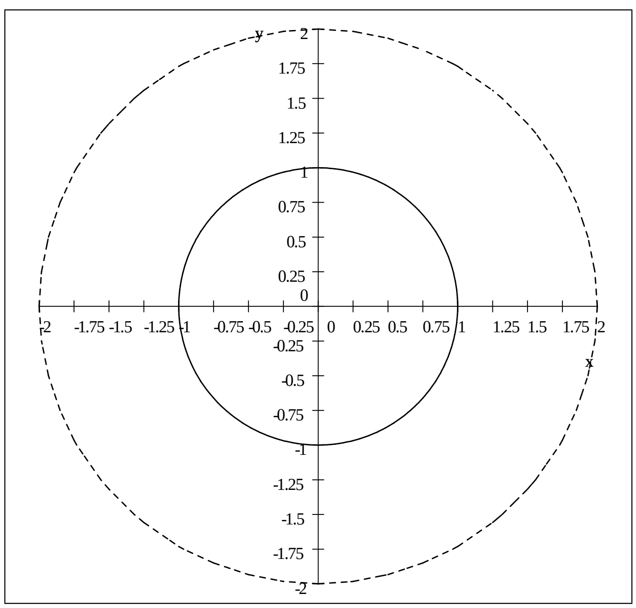
$$0\leq\left|r\left(\cos^3\theta+\sin^3\theta\right)\right|\leq 2\left|r\right|,$$

and since the leftmost limit and the rightmost limit are zero, it follows from the squeezing theorem that $\lim_{r\to0} r\left(\cos^3\theta+\sin^3\theta\right)=0.$

2. (a) Find and sketch the domain of the function $f(x,y)=\sqrt{x^2+y^2-1}+\ln\left(4-x^2-y^2\right)$.

Solution:

The domain of f contains the points (x,y) such that $x^2+y^2-1\geq0$ and $4-x^2-y^2>0$. The inequality $x^2+y^2-1\geq0$ gives all points outside or on the circle $x^2+y^2=1$. The inequality $4-x^2-y^2>0$ gives all points strictly inside the circle $x^2+y^2=4$. The intersection of these two sets give all points outside or on the circle $x^2+y^2=1$



but stricley inside the circle $x^2+y^2=4$.

(b) Find $h(x,y)=g(f(x,y))$ where $g(t)=t^2+\sqrt{t}$ and $f(x,y)=2x-3y-6$.

Solution:

$h(x,y)=(f(x,y))^2+\sqrt{f(x,y)}=(2x-3y-6)^2+\sqrt{2x-3y-6}.$

3. (a) Describe and sketch the graph of the surface $r = 2\cos\theta$.

Solution:

Since the equation $r = 2\cos\theta$ represents a circle in the xy-plane centered at $(1, 0)$ and with diameter 2, and since z is missing from the equation, the suface is a circular cylinder with directrix along the z-axis

(b) Write the equation $z = x^2 - y^2$ (a) in cylindrical coordinates and (b) in spherical coordinates.

Solution:

In cylindrical coordinates,

$$\begin{aligned} z &= r^2 \left(\cos^2\theta - \sin^2\theta\right) \\ &= r^2 \cos 2\theta. \end{aligned}$$

In spherical coordinates (since $z = \rho\cos\varphi$ and $r = \rho\sin\varphi$),

$$\begin{aligned} \rho\cos\varphi &= \rho^2 \sin^2\varphi \cos 2\theta \\ \rho &= \cot\theta \csc\theta \sec 2\theta. \end{aligned}$$

4. (a) Find the equation of the plane that passes through the line of intersection of the two planes $x - z = 1$ and $y + 2z = 3$ and is perpincicular to the plane $x + y - 2z = 1$.

Solution:

Put $z = 0$ and $z = -1$ in the equations of the two planes $x - z = 1$ and $y + 2z = 3$ to get two points on the line of intersection. This yeilds the two points $P(1, 3, 0)$ and $Q(0, 5, -1)$. Thus the vector $\overrightarrow{PQ} = \langle 1, -2, 1\rangle$ is parallel to our plane. Since our plane is also perpindicular to the plane $x + y - 2z = 1$, the vector $\langle 1, 1, -2\rangle$ is also parallel to our plane. Therefore, the normal to our plane is given by

$$\begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{vmatrix} = \langle 3, 3, 3\rangle.$$

The equation of our plane is

$$3(x - 1) + 3(y - 3) + 3z = 0$$

which simplifies to

$$x + y + z = 4.$$

(b) Determine whether the function $u = \ln\sqrt{x^2 + y^2}$ is a solution of the equation $u_{xx} + u_{yy} = 0$.

Solution:

$$\begin{aligned} u &= \frac{1}{2}\ln\left(x^2 + y^2\right), \\ u_x &= \frac{x}{(x^2 + y^2)},\ u_y = \frac{y}{(x^2 + y^2)}, \\ u_{xx} &= \frac{y^2 - x^2}{(x^2 + y^2)},\ u_{yy} = \frac{x^2 - y^2}{(x^2 + y^2)} \\ u_{xx} + u_{yy} &= \frac{y^2 - x^2}{(x^2 + y^2)} + \frac{x^2 - y^2}{(x^2 + y^2)} = 0. \end{aligned}$$