1. (a) Find the center and radius of the sphere $S : x^2 + y^2 + z^2 + 4x - 8y - 2z + 5 = 0$.

   Solution: Complete the squares to get $(x+2)^2 + (y-4)^2 + (z-1)^2 = 16$. Center: $(-2, 4, 1)$, Radius: 4

   (b) Find points A, B on the sphere S such that AB is a diagonal of the sphere.

   Solution: one way to obtain such points is $(-2-4, 4, 1), (-2+4, 4, 1)$. This gives $(-6, 4, 1), (2, 4, 1)$.

   (c) Find the distance from the origin to the sphere S.

   Solution: Distance = distance between the origin and the center − radius of the sphere $= \sqrt{4+16+1} - 4 = \sqrt{21} - 4$.

2. (a) Find the unit vector $\mathbf{u}$ with direction angles $\frac{1}{3}\pi, \frac{1}{4}\pi, \gamma$, where γ is an obtuse angle.

   Solution: To find γ, we use the formula $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, with $\alpha = \frac{1}{3}\pi, \beta = \frac{1}{4}\pi$. This gives $\cos^2\gamma = \frac{1}{4}$. Therefore, $\cos\gamma = -\frac{1}{2}$ (since γ is obtuse).

$$\begin{aligned} \mathbf{u} &= \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k} \\ &= \frac{1}{2}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} - \frac{1}{2}\mathbf{k}. \end{aligned}$$

   (b) Find the angle between the vectors the vectors $3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ to the nearest hundredth radians.

   Solution: $\cos\theta = \frac{11}{\sqrt{14}\sqrt{14}} = \frac{11}{14}$. $\theta = \cos^{-1}\frac{11}{14} = 0.67$.

   (c) **(2 points)** Find the vector projection of $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ onto the vector $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

   Solution: $\text{Proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b} = \frac{11}{14}(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = \frac{11}{14}\mathbf{i} + \frac{11}{7}\mathbf{j} - \frac{33}{14}\mathbf{k}$.

   (d) Show that if $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{c}$, then $\mathbf{b}$ and $\mathbf{c}$ have the same vector projction onto $\mathbf{a}$.

   Solution: $\text{Proj}_{\mathbf{a}}\mathbf{c} = \frac{\mathbf{a}\cdot\mathbf{c}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a} = \text{Proj}_{\mathbf{a}}\mathbf{b}$. Therefore, $\mathbf{b}$ and $\mathbf{c}$ have the same vector projction onto $\mathbf{a}$.

3. (a) Find the volume of the parallelepiped with edges along the vectors $2\mathbf{i} + \mathbf{j}$, $3\mathbf{i} - 2\mathbf{k}$, $3\mathbf{j} + 2\mathbf{k}$.

   Solution:

$$\begin{vmatrix} 2 & 1 & 0 \\ 3 & 0 & -2 \\ 0 & 3 & 2 \end{vmatrix} = 6$$

   Therefore, the volume of the parallelepiped $= |6| = 6$.

   (b) Find the distance between the point $P(1, 0, -1)$ and line through the points $A(1, 2, 1)$, $B(2, 2, -2)$.

Solution:

$$D = \frac{\left\| \overrightarrow{PA} \times \overrightarrow{AB} \right\|}{\left\| \overrightarrow{AB} \right\|}.$$

$$\begin{aligned} \overrightarrow{PA} &= \langle 0, 2, 2 \rangle, \\ \overrightarrow{AB} &= \langle 1, 0, -3 \rangle \\ \overrightarrow{PA} \times \overrightarrow{AB} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 2 \\ 1 & 0 & 3 \end{vmatrix} = 6\mathbf{i}+2\mathbf{j} - 2\mathbf{k}. \end{aligned}$$

Thus

$$D = \frac{\sqrt{44}}{\sqrt{10}} = \sqrt{4.4}.$$

4. (a) Determine whether $l_1 : x = 3 + t, y = 1 - t, z = 5 + 2t$ and $l_2 : x = 1, y = 4 - t, z = 9 - 2t$ intersect. If so, find their point of intersection.

   Solution: The system

   $$\begin{aligned} 3 + t_1 &= 1, \\ 1 - t_1 &= 4 - t_2 \end{aligned}$$

   has the solution

   $$t_1 = -2, \; t_2 = 1.$$

   we check the z values to get $5 + 2t_1 = 1, \;\; 9 - 2t_2 = 7$. Since these two values are not the same, the two lines do not intersect.

   (b) Find parametric equations of the line through $P(1, 4, -3)$ and perpindicular to the yz-plane.

   Solution: Direction of the line is $\langle 1, 0, 0 \rangle$. Therefore, the equatoins of the line are

   $$x = 1 + t, \; y = 4, \; z = -3.$$

5. (a) Find the equation of the plane through the point $P(2, 0, 1)$ and contains the line $l : x = 1 - 2t, \; y = 1 + 4t, \; z = 2 + t$.

   Solution: A point A on the line l is $(1, 1, 2)$. The vector $\overrightarrow{PA} = \langle -1, 1, 1 \rangle$ is in the plane. Also, the direction $\langle -2, 4, 1 \rangle$ of the line l is in the plane. Therefore, the normal to the plane is given by

   $$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 1 \\ -2 & 4 & 1 \end{vmatrix} = -3\mathbf{i} - \mathbf{j} - 2\mathbf{k}.$$

   Thus, the equaion of the plane is

   $$-3(x - 2) - y - 2(z - 1) = 0,$$

   or

   $$3x + y + 2z = 8.$$

(b) Find the distance from the point $P(2, -1, 3)$ to the plane $2x + 4y - z + 1 = 0$.

Solution:

$$D = \frac{|2(2) + 4(-1) - 3 + 1|}{\sqrt{4 + 16 + 1}} = \frac{2}{\sqrt{21}}.$$

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