Given the points $P\left(1,-1,2\right),\ Q(2,3,4),\ B(0,1,-1)$, find:

1. the equation of the sphere for which PQ is a diameter;

   Ans: Mid point between P and Q is $\left(\frac{1+2}{2},\frac{-1+3}{2},\frac{2+4}{2}\right)=\left(\frac{3}{2},1,3\right)$.

   Radius$=\frac{1}{2}\sqrt{(2-1)^2+(3+1)^2+(4-2)^2}=\frac{\sqrt{21}}{2}$. Equation of the sphere:

   $$\left(x-\frac{3}{2}\right)^2+(y-1)^2+(z-3)^2=\frac{21}{4}.$$

2. **(3 points)** the vectors $\overrightarrow{PB},\ \overrightarrow{PQ}$ and $\overrightarrow{PB}\times\ \overrightarrow{PQ}$;

   Ans: $\overrightarrow{PB}=\langle -1,2,-3\rangle,\ \overrightarrow{PQ}=\langle 1,4,2\rangle,\ \overrightarrow{PB}\times\ \overrightarrow{PQ}=\langle 16,-1,-6\rangle$

3. the direction cosines for $\overrightarrow{PB}$.

   Ans: $\left\|\overrightarrow{PB}\right\|=\sqrt{14}$. $\cos\alpha=-\frac{1}{\sqrt{14}},\ \cos\beta=\frac{2}{\sqrt{14}},\ \cos\gamma=-\frac{3}{\sqrt{14}}$.

4. **(3 points)** the degree measure of the acute angle $\widehat{BPQ}$;

   Ans: $\cos\theta=\frac{\left|\overrightarrow{PB}\cdot\overrightarrow{PQ}\right|}{\left\|\overrightarrow{PB}\right\|\left\|\overrightarrow{PQ}\right\|}=\frac{1}{\sqrt{14}\sqrt{21}}=\frac{1}{7\sqrt{6}}$. $\theta=86.57°$.

5. the area of the triangle BPQ;

   Ans: $\triangle PBQ=\frac{1}{2}\left\|\overrightarrow{PB}\times\ \overrightarrow{PQ}\right\|=\frac{1}{2}\sqrt{293}$.

6. the hight of the triangle BPQ when PQ is the base;

   Ans: $h=\frac{2\triangle PBQ}{\left\|\overrightarrow{PQ}\right\|}=\frac{\sqrt{293}}{\sqrt{21}}=3.74$

7. a decomposition of the vector $\overrightarrow{PB}$ into two vectors, one in the direction of $\overrightarrow{PQ}$ and the other orthogonal to it.

   Ans: Proj $_{\overrightarrow{PQ}}\overrightarrow{PB}=\frac{\overrightarrow{PB}\cdot\ \overrightarrow{PQ}}{\left\|\overrightarrow{PQ}\right\|^2}\ \overrightarrow{PQ}=\frac{1}{21}\langle 1,4,2\rangle=\left\langle\frac{1}{21},\frac{4}{21},\frac{2}{21}\right\rangle$.

   Component orthogonal to $\overrightarrow{PQ}=\overrightarrow{PB}-$Proj $_{\overrightarrow{PQ}}\overrightarrow{PB}=\langle -1,2,-3\rangle-\left\langle\frac{1}{21},\frac{4}{21},\frac{2}{21}\right\rangle=\left\langle -\frac{22}{21},\frac{38}{21},-\frac{65}{21}\right\rangle$.

8. parametric equations of the line that passes through P and Q;

   Ans: $x=1+t,\ y=-1+4t,\ z=2+2t$.

9. all points C on the line through P and Q such that the triangle PBC has a right angle;

   Ans: Since C is on PQ, $C=(1+t,-1+4t,2+2t)$ and $\overrightarrow{BC}=\langle 1+t,-2+4t,3+2t\rangle$. We hae two possibilities: either angle $C=90°$, or angle $B=90°$. In the first case, $\overrightarrow{BC} \cdot \overrightarrow{PQ} = 0$. This gives

$$\begin{aligned}
\langle 1+t, -2+4t, 3+2t \rangle \cdot \left\langle \frac{1}{21}, \frac{4}{21}, \frac{2}{21} \right\rangle &= 0 \\
-1+21t &= 0 \\
t &= \frac{1}{21} \\
C &= \left(\frac{22}{21}, -\frac{17}{21}, \frac{44}{21} \right).
\end{aligned}$$

In the second case, $\overrightarrow{BC} \cdot \overrightarrow{PB} = 0$. This gives

$$\begin{aligned}
\langle 1+t, -2+4t, 3+2t \rangle \cdot \langle -1, 2, -3 \rangle &= 0 \\
-14+t &= 0 \\
t &= 14 \\
C &= (15, 55, 30).
\end{aligned}$$

10. the distance between the point B and the line through P and Q;

    Ans: You may notice that the answer is the same as problem 6. Alternatively, you may use problem 9 to compute $\left\| \overrightarrow{BC} \right\|$ at $t = \frac{1}{21}$. In either case you get 3.74.