1. (a) If $f(x,y) = x^2 - 5y, h(t) = t^2$ and $g(x,y) = 5x - y^2$, compute $f(h(2), g(1,1))$ and $g(h(2), f(1,1))$

Solution:

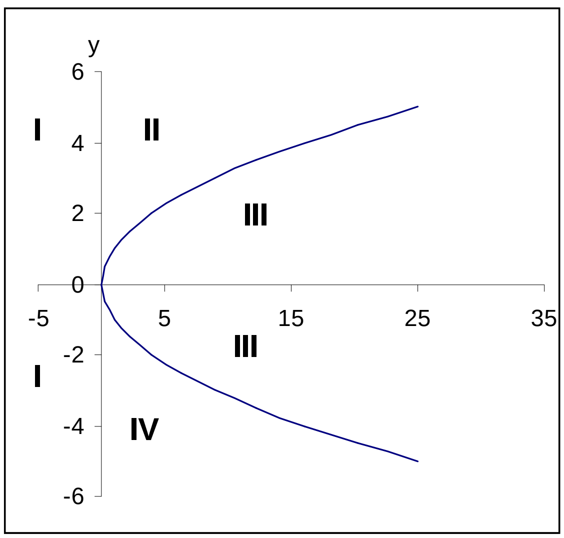
$h(2) = 4,\ g(1,1) = 5-1 = 4,\ f(h(2), g(1,1)) = f(4,4) = 16-20 = -4.$

$f(1,1) = -4.\ g(h(2), f(1,1)) = 20 - 16 = 4.$

(b) Sketch and shade the domain of the function $f(x,y) = \sqrt{x(y^2 - x)}$. Use dotted lines to indicate portions of the boundary that are not included and solid lines to indicate portions of the boundary that are included.

Solution:

The domain consists of all points (x,y) such that $x\left(y^2 - x\right) \geq 0$. The equation $x\left(y^2 - x\right) = 0$ gives $x = 0$ or $\left(y^2 - x\right) = 0$. The first of these two equations represents the y-axis. The second represents the parabola $x = y^2$. The two curves divide the plane into the 4 regions labled I, II, III, IV below. checking the inequality with sample points in each region shows that the inequality is satisfied by all points in regions II, IV. Thus the solution set consists of the regions II, IV, the y-axis and the parabola $x = y^2$.



2. (a) Compute

$$\lim_{(x,y)\to(0,0)} \frac{\tan 2\left(x^2 + y^2\right) + 3\sin\left(x^2 + y^2\right)}{\left(x^2 + y^2\right)}.$$

Solution:

Let $z = x^2 + y^2$. The problem is transformed into

$$\begin{aligned} & \lim_{z \to 0} \frac{\tan 2z + 3 \sin z}{z} \\ = \quad & \lim_{z \to 0} \frac{2 \sec^2 z + 3 \cos z}{1} \text{ (by L'hospital's rule)} \\ = \quad & 2 + 3 = 5. \end{aligned}$$

(b) Show that

$$\lim_{(x,y) \to (1,2)} \frac{y-2}{x-1}$$

does not exist.

Solution:

a) Approach the point $(1, 2)$ through the horizontal line $y = 2$. We get the limit problem

$$\lim_{x \to 1} \frac{0}{x-1} = 0.$$

b) Approach the point $(1, 2)$ through the line $y = 2x$. We get the limit problem

$$\lim_{x \to 1} \frac{2x-2}{x-1} = 2.$$

Since we obtain two different limits, the limit does not exist.

3. (a) Find a point P at which the function $f(x, y) = x^2 y$ has a local linear approximation $L(x, y) = 4y - 4x + 8$.

Solution:

$f_x = 2xy$, $f_y = x^2$. At the point P, $f_x = -4$, $f_y = 4$. Therefore, $2xy = -4$, $x^2 = 4$. Solving the second equation, we get $x = \pm 2$. Substituting in the first, we get $y = \mp 1$. We have two possible solutions: $(2, -1)$ and $(-2, 1)$. Computing the local linear approximation at $(2, -1)$ we get $L(x, y) = 4y - 4x + 8$. Computing the local linear approximation at $(-2, 1)$ we get $L(x, y) = 4y - 4x - 8$. Thus P is the point $(2, -1)$.

(b) Determine dw for $w = \sqrt{x} + \sqrt{y} + \sqrt{z}$.

Solution:

$dw = w_x dx + w_y dy + w_z dz$. Thus, $dw = \frac{1}{2\sqrt{x}} dx + \frac{1}{2\sqrt{y}} dy + \frac{1}{2\sqrt{z}} w_z dz$.

4. (a) Suppose $w = xy + yz$, $y = \sin x$, $z = e^x$. Use a chain rule to find $\frac{dw}{dx}$.

Solution:

$$\begin{aligned} \frac{dw}{dx} &= \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x} \\ &= y + (x+z)\cos x + ye^x. \end{aligned}$$

(b) Find $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$ for $ye^x - 5\sin 3z = 3z$.

Solution:

Let $f(x,y,z) = ye^x - 5\sin 3z - 3z$. By implicit differentiation,

$$\begin{aligned} \frac{\partial z}{\partial x} &= -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} = \frac{ye^x}{3+15\cos 3z}, \\ \frac{\partial z}{\partial y} &= -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}} = \frac{e^x}{3+15\cos 3z}. \end{aligned}$$

5. (a) Given that $f_x(-5,1) = -3,\ f_y(-5,1) = 2$, find the directional derivative of f at the point $P(-5,1)$ in the direction from P to $Q(-4,3)$.

Solution:

$\overrightarrow{PQ} = \langle 1,2\rangle$. $\mathbf{u} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$. $D_{\mathbf{u}}f = \nabla f.\mathbf{u} = \langle -3,2\rangle \cdot \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle = \frac{1}{\sqrt{5}}$.

(b) Find a unit vector in the direction in which the functions $f(x,y) = 4e^{xy}\sin z$ decreases most rapidly at the point $P\left(0,1,\frac{\pi}{3}\right)$ and find the rate of change of f at P in that direction.

Solution:

$\nabla f = \langle 4ye^{xy}\sin z, 4xe^{xy}\sin z, 4e^{xy}\cos z\rangle$. $\nabla f(P) = \left\langle 2\sqrt{3}, 0, 2\right\rangle$. $\|\nabla f(P)\| = 4$. Thus, f decreases most rapidly in the direction os the vector $-\nabla f(P) / \|\nabla f(P)\| = \left\langle -\frac{\sqrt{3}}{2}, 0, \frac{1}{2} \right\rangle$. The rate of change of f in that direction is $-\|\nabla f(P)\| = -4$.