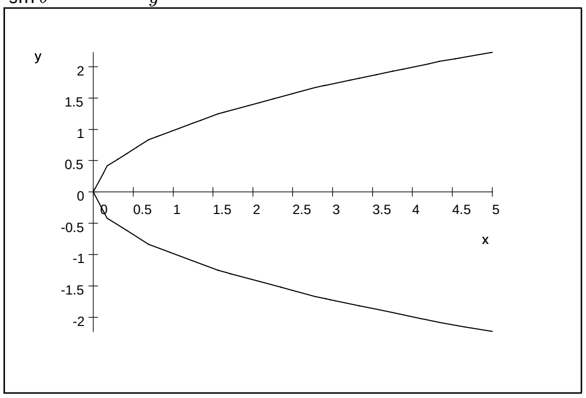
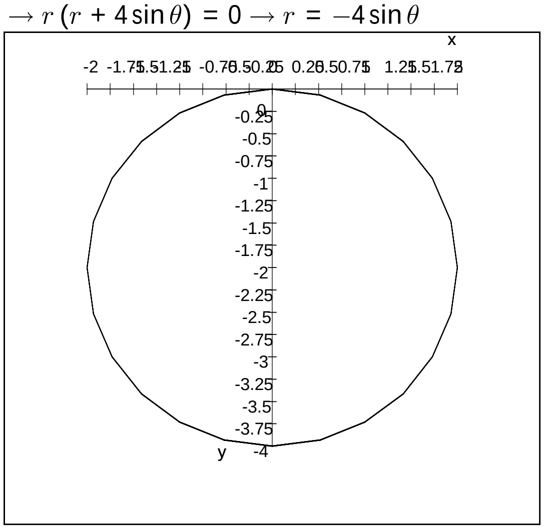
1. (a) Transform the equation $r = \csc\theta\cot\theta$ into rectangular coordinates and sketch its graph.

$r\sin\theta = \frac{r\cos\theta}{r\sin\theta} \rightarrow y = \frac{x}{y} \rightarrow y^2 = x.$



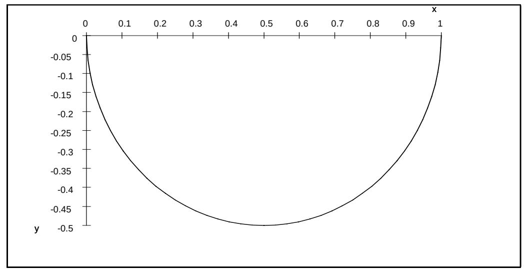
(b) (3 points) Transform the polar equatoin $x^2 + y^2 + 4y = 0$ into a polar equation of the form $r = f(\theta)$ and sketch its graph.

$r^2 + 4r\sin\theta = 0 \rightarrow r(r + 4\sin\theta) = 0 \rightarrow r = -4\sin\theta$

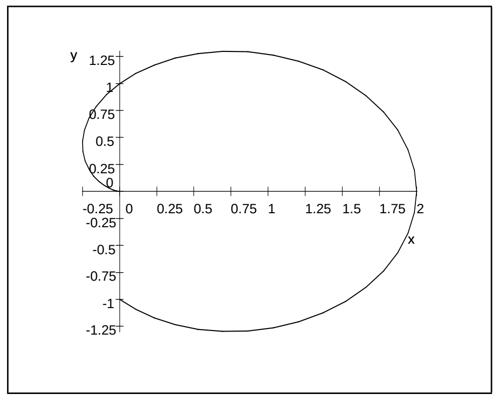


(c) Sketch the graphs of the following polar equaions

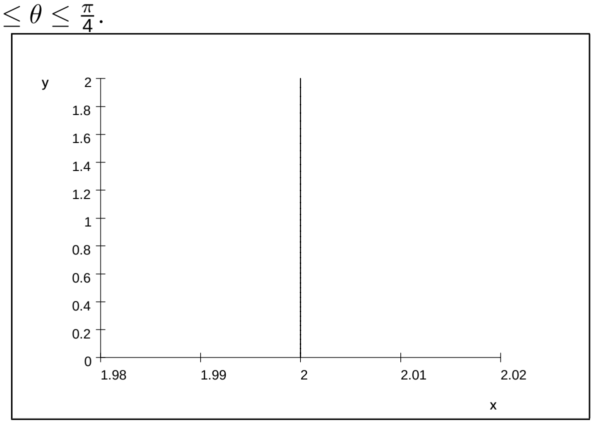
i. $r = \cos\theta, \ \frac{\pi}{2} \leq \theta \leq \pi$



ii. $r = 1 + \cos\theta,\ -\frac{\pi}{2} \le \theta \le \pi$



iii. $r = 2\sec\theta,\ 0 \le \theta \le \frac{\pi}{4}$.



2. (a) Find $\frac{dy}{dx}$ at $t = \frac{\pi}{6}$ for the parametric curve $x = t + \cos t,\ y = 1 - \sin t$.
$\frac{dy}{dt} = -\cos t, \frac{dx}{dt} = 1 - \sin t \rightarrow \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{-\cos t}{1-\sin t}$. At $t = \frac{\pi}{6}$, $\frac{dy}{dx} = \frac{-\cos\frac{\pi}{6}}{1-\sin\frac{\pi}{6}} = -\sqrt{3}$.

(b) (2 points each)

i. Show that the arc length of the cardioid $r = 1 + \cos\theta$ is given by

$L = \sqrt{2}\int_0^{2\pi}\sqrt{1+\cos\theta}\,d\theta.$

$$\begin{aligned} L &= \int_0^{2\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_0^{2\pi}\sqrt{(1+\cos\theta)^2+\sin^2\theta}\,d\theta \\ &= \int_0^{2\pi}\sqrt{1+2\cos\theta+\cos^2\theta+\sin^2\theta}\,d\theta = \int_0^{2\pi}\sqrt{2+2\cos\theta}\,d\theta \\ &= \sqrt{2}\int_0^{2\pi}\sqrt{1+\cos\theta}\,d\theta. \end{aligned}$$

ii. Using the identity $1+\cos\theta = 2\cos^2\frac{\theta}{2}$ show how to compute the arc length of the cardioid.

$$\begin{aligned} L &= \sqrt{2}\int_0^{2\pi}\sqrt{1+\cos\theta}\,d\theta = \sqrt{2}\int_0^{2\pi}\sqrt{2\cos^2\frac{\theta}{2}}\,d\theta \\ &= 2\int_0^{2\pi}\left|\cos\frac{\theta}{2}\right|d\theta = 4\int_0^{\pi}\cos\frac{\theta}{2}\,d\theta = 8. \end{aligned}$$

(c) (4 points) Find the equation of the tangent line to the graph of the curve $r = \frac{1}{\theta}$ at $\theta = \frac{\pi}{2}$.

$$\frac{dy}{dx} = \frac{\frac{1}{\theta}\cos\theta + \frac{1}{\theta^2}\sin\theta}{\frac{1}{\theta}\sin\theta - \frac{1}{\theta^2}\cos\theta}$$

At $\theta = \frac{\pi}{2}$, $r = \frac{2}{\pi}$, $\frac{dy}{dx} = \frac{2}{\pi}$. The point $\left(\frac{2}{\pi}, \frac{\pi}{2}\right)$ in polar coordinates corresponds to $\left(0, \frac{2}{\pi}\right)$ in rectangular coordinates. Thus the equation of the tangent is

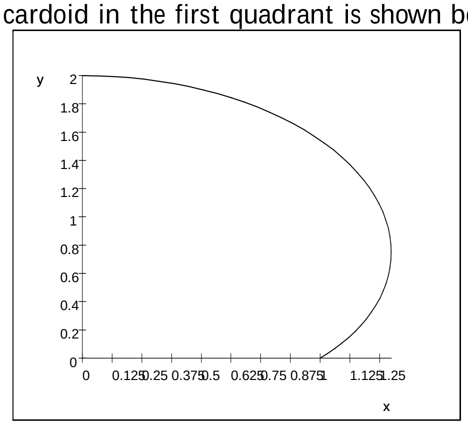
$$y - \frac{2}{\pi} = \frac{2}{\pi}x$$

or

$$\pi y - 2x = 2.$$

3. (a) (2 points) Set up an integral to compute the area of the region in the first quadrant within the cardioid $r = 1+\sin\theta$.

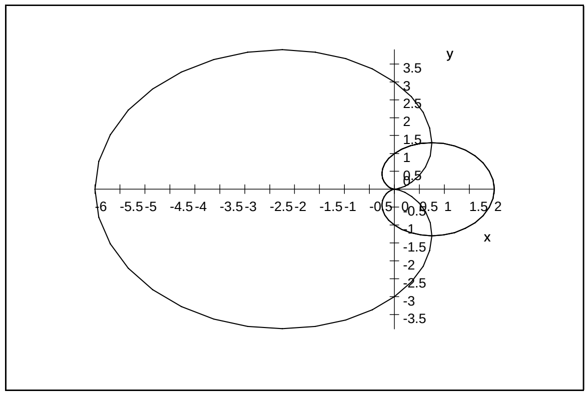
The portion of the cardoid in the first quadrant is shown below.



$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\sin\theta)^2\, d\theta.$$

(b) (3 points each)

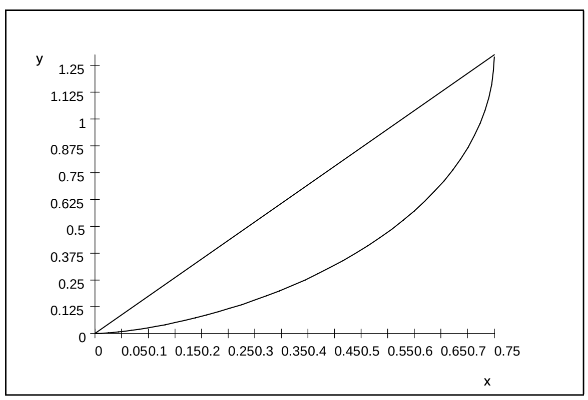
i. Find all points of intersection of the two cardioids $r = 1 + \cos\theta$ and $r = 3(1-\cos\theta)$.
Solving the two equations simultanuously,



$$1+\cos\theta = 3(1-\cos\theta) \rightarrow 4\cos\theta = 2 \rightarrow \cos\theta = \frac{1}{2} \rightarrow \theta = -\frac{\pi}{3}, \frac{\pi}{3} \rightarrow r = \frac{3}{2}$$

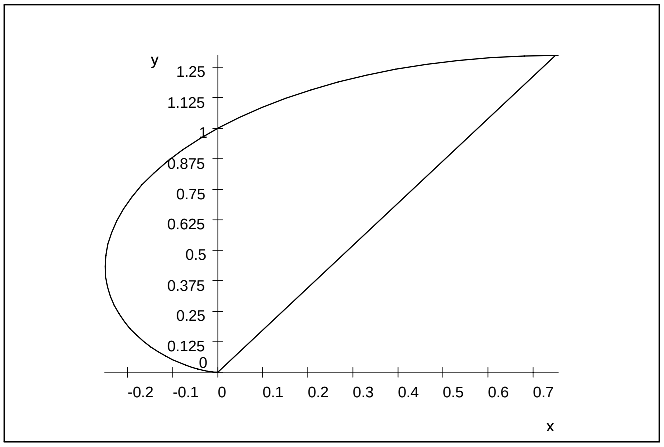
From the graph we also see that $(0, 0)$ is a point of intersecetion.
Thus the points of intersection are $(0, 0)$, $\left(\frac{3}{2}, -\frac{\pi}{3}\right)$, $\left(\frac{3}{2}, \frac{\pi}{3}\right)$.

ii. Give the common area between the two cardioids in terms of integrals.
From part i,



A1: between second cardioid and $\theta = \frac{\pi}{3}$

$$A1 = \frac{9}{2}\int_0^{\frac{\pi}{3}} (1 - \cos\theta)^2 \, d\theta$$



A2: Between first cardioid and $\theta = \frac{\pi}{3}$, $\theta = \pi$.

$$A2 = \frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} (1 + \cos\theta)^2 \, d\theta$$

$$A = 2\,(A1 + A2)$$