1 Spaces with differentiable norms

Recall that a function $f : X \to \mathbb{R}$ is called Fréchet differentiable at $x \in X$ if there exists a continuous linear functional, denoted $f'(x)$, such that

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - \langle f'(x), h \rangle}{\|h\|} = 0. \tag{1}$$

Definition 1 *A Banach space X is called Fréchet smooth if for each $0 \neq x \in X$ there exists an $x^* \in X^*$ (that may depend on x) such that*

$$\lim_{h \to 0} \frac{\|x+h\| - \|x\| - \langle x^*, h \rangle}{\|h\|} = 0. \tag{2}$$

- The norm on X cannot be differentiable at 0. This means that the limit (2) cannot exist at 0, for otherwise we have

  $$\lim_{h \to 0} \frac{\|h\| - \langle x^*, h \rangle}{\|h\|} = 0,$$

  which means $x^* \neq 0$. But then taking a sequence $h_n \to 0$ (and hence, $-h_n \to 0$) we get

  $$\left\langle x^*, \frac{h_n}{\|h_n\|} \right\rangle \to \pm 1$$

  which is impossible.

Lemma 2 *In any Banach space with norm $\|\cdot\|$, $\|\cdot\|^2$ is differerntiable at $x = 0$ with derivative* 0.

Proof. This follows immediately from (1) by subistituting $f(x) := \|x\|^2$ and $f'(0) = 0$. ■

Lemma 3 *Suppose X is Fréchet smooth. Let $0 \neq x \in X$ and let $x^* = \partial \|x\|$ Then $\|x^*\| = 1$.*

Proof. Let $\varepsilon > 0$. There exists a $\delta > 0$ such that $h \in \delta B \Longrightarrow$

$$\begin{aligned} |\langle x^*, h \rangle| &\leq |\|x+h\| - \|x\|| + \varepsilon \|h\| \\ &\leq \|h\| (1 + \varepsilon). \end{aligned}$$

Therefore,

$$\|x^*\| \leq 1.$$

On the other hand, taking $h = tx$ in (2) gives

$$1 - \left\langle x^*, \frac{x}{\|x\|} \right\rangle = 0.$$

Therefore,

$$\|x^*\| \geq 1.$$

Hence, $\|x^*\| = 1$. ■

- Observe that the derivative functional x^* acheives its norm at the point $\frac{x}{\|x\|}$.

2 The subderivative of a norm

Theorem 4 *Let* $\overline{x} \in X$*. Then* $\partial \left\| \cdot - \overline{x} \right\| (\overline{x}) = \mathbf{B}^*$.

Proof. Since $x \mapsto \|x - \overline{x}\|$ is convex and continuous,

$$\partial \left\| \cdot - \overline{x} \right\| = \widehat{\partial} \left\| \cdot - \overline{x} \right\| .$$

Let $x^* \in \partial \left\| \cdot - \overline{x} \right\| (\overline{x})$. Then

$$\underline{\lim}_{x \to \overline{x}} \frac{\|x - \overline{x}\| - \langle x^*, x - \overline{x} \rangle}{\|x - \overline{x}\|} \geq 0.$$

It follows that for any $\varepsilon > 0$ ther exists a $\delta > 0$ such that for all $x \in \overline{x} + \delta \mathbf{B}$,

$$\frac{\|x - \overline{x}\| - \langle x^*, x - \overline{x} \rangle}{\|x - \overline{x}\|} \geq -\varepsilon.$$

Then

$$\left\langle x^*, \frac{x - \overline{x}}{\|x - \overline{x}\|} \right\rangle \leq 1 + \varepsilon.$$

Therefore,

$$\|x^*\| \leq 1 + \varepsilon$$

and since ε is arbitrary, $\|x^*\| \leq 1$.

On the other hand, assume that $x^* \in \mathbf{B}^*$. Then

$$\frac{\|x - \overline{x}\| - \langle x^*, x - \overline{x} \rangle}{\|x - \overline{x}\|} = 1 - \left\langle x^*, \frac{x - \overline{x}}{\|x - \overline{x}\|} \right\rangle \geq 1 - \|x^*\| \geq 0.$$

Therefore,

$$\underline{\lim}_{x \to \overline{x}} \frac{\|x - \overline{x}\| - \langle x^*, x - \overline{x} \rangle}{\|x - \overline{x}\|} \geq 0$$

and $x^* \in \partial \|x - \overline{x}\| (\overline{x})$. ■

Theorem 5 *Let* $0 \neq \overline{x} \in X$*. Then*

$$\phi \neq \partial \left\| \cdot \right\| (\overline{x}) = \left\{ x^* \in X^* : \|x^*\| = 1, \langle x^*, \overline{x} \rangle = \|\overline{x}\| \right\} .$$

Proof. Since $x \mapsto \|x\|$ is convex and continuous,

$$\partial \left\| \cdot \right\| = \widehat{\partial} \left\| \cdot \right\|$$

Let $0 \neq \overline{x} \in X$ and notice that, since epi $\left\| \cdot \right\|$ is convex,

$$\widehat{\partial} \left\| \cdot \right\| (\overline{x}) = \left\{ x^* \in X^* : \langle x^*, x - \overline{x} \rangle - (\|x\| - \|\overline{x}\|) \leq 0 \ \forall x \in X \right\} .$$

Let $x^* \in X^*$ be such that $\|x^*\| = 1$ and $\langle x^*, \overline{x} \rangle = \|\overline{x}\|$ (one exists by the Hahn-Banach theorem). Then, for any $x \in X$,

$$\langle x^*, x - \overline{x} \rangle - (\|x\| - \|\overline{x}\|) = \langle x^*, x \rangle - \|x\| \leq \|x^*\| \|x\| - \|x\| = 0.$$

Thus, $x^* \in \widehat{\partial} \|\cdot\| (\overline{x})$, $\partial \|\cdot\| (\overline{x}) \neq \phi$ and $\partial \|\cdot\| (\overline{x}) \subset \{x^* \in X^* : \|x^*\| = 1, \langle x^*, \overline{x} \rangle = \|\overline{x}\|\}$. On the other hand, assume $x^* \in \widehat{\partial} \|\cdot\| (\overline{x})$. Then

$$\langle x^*, x - \overline{x} \rangle - (\|x\| - \|\overline{x}\|) \leq 0 \ \forall x \in X$$

Rearranginge, we get

$$\begin{aligned} \langle x^*, x - \overline{x} \rangle &\leq \|x\| - \|\overline{x}\| \\ &\leq \|x - \overline{x}\| \ \forall x \in X, \end{aligned}$$

which means that $\|x^*\| \leq 1$. Furthermore, we then have

$$\langle x^*, x \rangle - \|x\| \leq \langle x^*, \overline{x} \rangle - \|\overline{x}\| \leq 0 \ \forall x \in X.$$

we claim that this implies that $\langle x^*, \overline{x} \rangle - \|\overline{x}\| = 0$. Otherwise, $\langle x^*, \overline{x} \rangle - \|\overline{x}\| < 0$ and

$$\langle x^*, x \rangle - \|x\| \leq \langle x^*, \overline{x} \rangle - \|\overline{x}\| < 0 \ \forall x \in X.$$

In particular, taking $x = \frac{1}{2}\overline{x}$ we get

$$\frac{1}{2} (\langle x^*, \overline{x} \rangle - \|\overline{x}\|) \leq \langle x^*, \overline{x} \rangle - \|\overline{x}\|$$

which result in the rediculus statment

$$\frac{1}{2} \geq 1.$$

Thus, $\langle x^*, \overline{x} \rangle = \|\overline{x}\|$ and $\|x^*\| = 1$. ■

Example This example illustrates cases where the norm subdifferential is a singleton and others where it is a whole set of points. Let $X = (\mathbb{R}^2; \|\cdot\|_\infty)$. Then $X^* = (\mathbb{R}^2; \|\cdot\|_1)$ and for $\eta = (\xi_1, \xi_2) \in X^*$ and $x = (x_1, x_2) \in X$, $\langle \eta, x \rangle = \xi_1 x_1 + \xi_2 x_2$. Let's compute $\partial \|\cdot\|_\infty (1, 1/2)$ and $\partial \|\cdot\|_\infty (1, 1)$. For the first subdifferential, we need to find all $\eta \in X^*$ with $\|\eta\|_1 = 1$ and $\langle \eta, (1, 1/2) \rangle = \|(1, 1/2)\|_\infty = 1$. Thus, we need to solve the silutanuous system

$$\begin{aligned} \xi_1 + 1/2\xi_2 &= 1, \\ |\xi_1| + |\xi_2| &= 1. \end{aligned}$$

It can easily be verified that the only solution to this system is $\eta = (1, 0)$. Therefore,

$$\partial \|\cdot\|_\infty (1, 1/2) = \{(1, 0)\}.$$

For the second subdifferential the corresponding system that needs to be solved is

$$\begin{aligned} \xi_1 + \xi_2 &= 1, \\ |\xi_1| + |\xi_2| &= 1. \end{aligned}$$

The set of solutions S in this case can be described as

$$S = \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2 : 0 \leq \xi_1, \xi_2 \leq 1, \xi_1 + \xi_2 = 1. \right\}$$

Therefore,

$$\partial \left\| \cdot \right\|_\infty (1, 1) = S.$$

Example This example illustrates that the norm subdifferetials at two close points may not be close. As in the previous example, let $X = \left(\mathbb{R}^2; \left\| \cdot \right\|_\infty\right)$, $X^* = \left(\mathbb{R}^2; \left\| \cdot \right\|_1\right)$. Let $a > 0$, $\varepsilon > 0$ be given. The two vctors $(a + \varepsilon, a)$ and $(a, a + \varepsilon)$ are such that $\left\| (a + \varepsilon, a) - (a, a + \varepsilon) \right\|_\infty = \left\| (\varepsilon, -\varepsilon) \right\|_\infty = \varepsilon$. On the other hand, $\partial \left\| \cdot \right\|_\infty (a + \varepsilon, a) = (1, 0)$ while $\partial \left\| \cdot \right\|_\infty (a, a + \varepsilon) = (0, 1)$. In this case $\left\| (1, 0) - (0, 1) \right\|_1 = 2$.