Extremal Principle in Variational Analysis

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1 (2.2.1) Approximate Extremal Principle in Smooth Banach Spaces

Theorem 1 *(2.10 approximate extremal principle in Fréchet smooth Banach Spaces)*
The approximate extremal principle is valid in any Banach space X with Fréchet differentiable norm.

Proof. We suppose that $\{\Omega_1, \Omega_2, \ldots, \Omega_n; \overline{x}\}$ is an extremal system in X. Let $\Omega = \Omega_2 \times \Omega_3 \times \ldots \times \Omega_n$. Vectors $(x, u) \in \Omega_1 \times \Omega$ will be denoted by z and x_D will denote the $(n-1)$-tuple $(x, x, \ldots, x)$. Assume $0 < \varepsilon < \frac{1}{2}$. Choose $a \in \frac{\varepsilon^3}{16}\mathbf{B}^{n-1}$ such that

$$x_D + a \notin \Omega \ \forall x \in \Omega_1$$

Let $\varepsilon_k \searrow 0$ such that $\varepsilon_1 = \frac{\varepsilon^3}{16}$ and $\varepsilon_{k-2} \geq \varepsilon_{k-1} + \varepsilon_k$ (a particular choice of ε_k wil be made later). Recursively define the sequences $\{\varphi_k\}_{k=0}^{\infty}, \{W_k\}_{k=0}^{\infty}, \{z_k\}_{k=1}^{\infty}$ as follows:

For $k = 0$;

$$\begin{aligned} \varphi_0(z) \quad &: \quad = \|x_D - u + a\|, \ z \in \Omega_1 \times \Omega, \\ W_0 \quad &= \quad \Omega_1 \times \Omega, \end{aligned}$$

where the norm in X^n is defined as $\|z\| = \sqrt{\|x\|^2 + \|u_2\|^2 + \|u_3\|^2 + \ldots + \|u_n\|^2}$.

Set

$$z_1 = (\overline{x}, \overline{x}_D).$$

For $k = 1, 2, \ldots$;

$$\begin{aligned} \varphi_k(z) \quad &: \quad = \varphi_{k-1}(z) + \frac{\varepsilon}{2^k}\|z - z_k\|^2, \\ W_k \quad &= \quad \left\{z \in \Omega_1 \times \Omega : \varphi_k(z) \leq \varphi_{k-1}(z_k) + \varepsilon_k\right\} \end{aligned}$$

and choose $z_{k+1} \in W_k$ such that

$$\varphi_k(z_{k+1}) < \inf_{W_k} \varphi_k(z) + \varepsilon_k$$

- $z_1 \in W_1$ and $\operatorname{diam} W_1 < \varepsilon$: For the first part,

$$\varphi_1(z_1) = \varphi_0(z_1) + \frac{\varepsilon}{2}\|z_1 - z_1\|^2 < \varphi_0(z_1) + \varepsilon_1.$$

  For the second part, let $z \in W_1$. Then

$$\begin{aligned}
\varphi_1(z) &\leq \varphi_0(z_1) + \varepsilon_1 = \|a\| + \frac{\varepsilon^3}{16} < \frac{\varepsilon^3}{8}; \\
\varphi_0(z) + \frac{\varepsilon}{2}\|z - z_1\|^2 &< \frac{\varepsilon^3}{8}; \\
\|z - z_1\|^2 &< \frac{\varepsilon^2}{4}.
\end{aligned}$$

- $\{W_k\}_{k=0}^{\infty}$ is a decreasing sequence: Let $z \in W_k$. Then

$$\begin{aligned}
\varphi_k(z) &\leq \varphi_{k-1}(z_k) + \varepsilon_k < \varphi_{k-1}(u) + (\varepsilon_{k-1} + \varepsilon_k)\,\forall u \in W_{k-2} \\
&= \varphi_{k-2}(u) + \frac{\varepsilon}{2^{k-1}}\|u - z_{k-1}\|^2 + (\varepsilon_{k-1} + \varepsilon_k).
\end{aligned}$$

  Since $z_{k-1} \in W_{k-2}$,

$$\begin{aligned}
\varphi_k(z) &< \varphi_{k-2}(z_{k-1}) + (\varepsilon_{k-1} + \varepsilon_k) \\
&\leq \varphi_{k-2}(z_{k-1}) + \varepsilon_{k-2}.
\end{aligned}$$

  Thus

$$\varphi_{k-1}(z) + \frac{\varepsilon}{2^{k-1}}\|z - z_k\|^2 \leq \varphi_{k-2}(z_{k-1}) + \varepsilon_{k-2}$$

  which gives

$$\varphi_{k-1}(z) \leq \varphi_{k-2}(z_{k-1}) + \varepsilon_{k-2}$$

  and $z \in W_{k-1}$.

- $\operatorname{diam} W_k \to 0$: Let $z \in W_k$. Then

$$\begin{aligned}
\varphi_k(z) &\leq \varphi_{k-1}(z_k) + \varepsilon_k \\
\varphi_{k-1}(z) + \frac{\varepsilon}{2^{k-1}}\|z - z_k\|^2 &\leq \varphi_{k-1}(z_k) + \varepsilon_k \leq \varphi_{k-1}(z) + (\varepsilon_{k-1} + \varepsilon_k) \\
\|z - z_k\|^2 &\leq \frac{2^{k-1}(\varepsilon_{k-1} + \varepsilon_k)}{\varepsilon}.
\end{aligned}$$

  If we choose

$$\varepsilon_k = \frac{\varepsilon_1}{2^{2k+1}},\ k > 1$$

  then $\|z - z_k\| \to 0$ as $k \to \infty$.

- $\bigcap W_k = \{\overline{z}\}$: This follows immediately from the completeness of X.

Define the function $\varphi : \Omega_1 \times \Omega \to \mathbb{R}$ by

$$\varphi(z) = \lim_{k\to\infty} \varphi_k(z) = \varphi_0(z) + \varepsilon \sum_{k=1}^{\infty} \frac{\|z - z_k\|^2}{2^k}$$

- $\overline{z}$ is a minimizer of φ over $\Omega_1 \times \Omega$:
  - We show first that $\varphi_k(z_{k+1}) \to \varphi(\overline{z})$:

$$\begin{aligned} |\varphi_k(z_{k+1}) - \varphi_k(\overline{z})| &\leq |\varphi_0(z_{k+1}) - \varphi_0(\overline{z})| + \varepsilon \sum_{j=1}^{k} \frac{\left| \|z_{k+1} - z_j\|^2 - \|\overline{z} - z_j\|^2 \right|}{2^j} \\ &= |\varphi_0(z_{k+1}) - \varphi_0(\overline{z})| \\ &\quad + \varepsilon \sum_{j=1}^{k} \frac{|(\|z_{k+1} - z_j\| - \|\overline{z} - z_j\|)| (\|z_{k+1} - z_j\| + \|\overline{z} - z_j\|)}{2^j} \\ &\leq |\varphi_0(z_{k+1}) - \varphi_0(\overline{z})| + \varepsilon \sum_{j=1}^{k} \frac{2\delta_j \|z_{k+1} - \overline{z}\|}{2^j} \\ &\leq |\varphi_0(z_{k+1}) - \varphi_0(\overline{z})| + M\varepsilon \|z_{k+1} - \overline{z}\|, \end{aligned}$$

    where

$$M = \sum_{j=1}^{\infty} \frac{\delta_j}{2^{j-1}}$$

    and

$$\delta_j = \operatorname{diam} W_j.$$

    Since $z_{k+1} \to \overline{z}$ and φ_0 is continuous, $|\varphi_k(z_{k+1}) - \varphi_k(\overline{z})| \to 0$ as $k \to \infty$.
    Now

$$|\varphi_k(z_{k+1}) - \varphi(\overline{z})| \leq |\varphi_k(z_{k+1}) - \varphi_k(\overline{z})| + |\varphi_k(\overline{z}) - \varphi(\overline{z})| \to 0 \text{ as } k \to \infty.$$

  - Next, we show that $\varphi(z) \geq \varphi(\overline{z})$ for all $z \in X \times \Omega$: For $z \in X \times \Omega$, and $z \neq \overline{z}$, $z \notin W_k$ for all k sufficiently large. Then

$$\varphi_{k+1}(z) > \varphi_k(z_{k+1}) + \varepsilon_k.$$

    Taking the limit as $k \to \infty$,

$$\varphi(z) \geq \varphi(\overline{z}).$$

It follows that the function

$$\psi(z) := \varphi(z) + \delta(z; \Omega_1 \times \Omega)$$

achieves a minimum over X^n at $\overline{z}$. Thus

$$0 \in \widehat{\partial}\psi(\overline{z}) = \varphi'(\overline{z}) + \widehat{\partial}\delta(\overline{z}; \Omega_1 \times \Omega)$$

(recall that $\varphi(z)$ is differetiable at $\overline{z}$ since $\varphi(\overline{z}) > 0$).

$$
\begin{aligned}
-\varphi'(\overline{z}) &\in \widehat{N}(\overline{z};\Omega_1\times\Omega) = \widehat{N}(\widetilde{x};\Omega_1)\times\widehat{N}(\widetilde{u};\Omega) \\
&= \widehat{N}(\widetilde{x};\Omega_1)\times\prod_{i=2}^{n}\widehat{N}(\widetilde{u}_i;\Omega_i),
\end{aligned}
$$

where we put $\overline{z} = (\widetilde{x},\widetilde{u})$. Thus

$$
\frac{\partial\varphi}{\partial x}|_{\overline{z}} \in \widehat{N}(\widetilde{x};\Omega_1), \qquad \frac{\partial\varphi}{\partial u}|_{\overline{z}} \in \widehat{N}(\widetilde{u};\Omega).
$$

Now

$$
\begin{aligned}
\frac{\partial\varphi}{\partial x}|_{\overline{z}} &= \frac{\partial\varphi_0}{\partial x}|_{\overline{z}} + \varepsilon\sum_{k=1}^{\infty}\frac{\|\widetilde{x}-x_k\|}{2^{j-1}}D(\widetilde{x}-x_k), \\
\frac{\partial\varphi}{\partial u_i}|_{\overline{z}} &= \frac{\partial\varphi_0}{\partial x}|_{\overline{z}} + \varepsilon\sum_{k=1}^{\infty}\frac{\|\widetilde{u}_i-u_{ik}\|}{2^{j-1}}D(\widetilde{u}_i-u_{ik}),\ 2\le i\le n,
\end{aligned}
$$

where $D(\cdot)$ denotes the derivative of the norm in X. Since $\|\overline{z}-z_k\| \le \operatorname{diam} W_k \le \operatorname{diam} W_1 < \varepsilon$, $\varepsilon < 1/2$ and $\|D(\cdot)\| = 1$,

$$
\begin{aligned}
\varepsilon\sum_{k=1}^{\infty}\frac{\|\widetilde{x}_i-x_k\|}{2^{j-1}}D(\widetilde{u}_i-u_{ik}) &\le 2\varepsilon^2 < \varepsilon \\
\varepsilon\sum_{k=1}^{\infty}\frac{\|\widetilde{u}_i-u_{ik}\|}{2^{j-1}}D(\widetilde{u}_i-u_{ik}) &\le 2\varepsilon^2 < \varepsilon,\ 2\le i\le n.
\end{aligned}
$$

We proceed by computing $\frac{\partial\varphi_0}{\partial x}|_{\overline{z}}$, $\frac{\partial\varphi_0}{\partial u}|_{\overline{z}}$. Since

$$
\begin{aligned}
\varphi_0(\overline{z}) &= \|\widetilde{x}-\widetilde{u}+a\| = \sqrt{\sum_{i=2}^{n}\|\widetilde{x}-\widetilde{u}_i+a_i\|^2}, \\
\frac{\partial\varphi_0}{\partial x}|_{\overline{z}} &= \frac{\sum_{i=2}^{n}\|\widetilde{x}-\widetilde{u}_i+a_i\|D(\widetilde{x}-\widetilde{u}_i+a_i)}{\|\widetilde{x}-\widetilde{u}+a\|}
\end{aligned}
$$

and

$$
\frac{\partial\varphi_0}{\partial u}|_{\overline{z}} = \left(-\frac{\|\widetilde{x}-\widetilde{u}_j+a_j\|D(\widetilde{x}-\widetilde{u}_j+a_j)}{\|\widetilde{x}-\widetilde{u}+a\|}\right)_{j=2}^{n}.
$$

Clearly

$$
\frac{\partial\varphi_0}{\partial x}|_{\overline{z}} + \sum_{i=2}^{n}\frac{\partial\varphi_0}{\partial u_j}|_{\overline{z}} = 0
$$

and

$$
\sum_{i=2}^{n}\left\|\frac{\partial\varphi_0}{\partial u_j}|_{\overline{z}}\right\|^2 = 1.
$$

Thus the functionals

$$x_1^* = \frac{\partial \varphi_0}{\partial x} \mid_{\overline{z}}, x_i^* = \frac{\partial \varphi_0}{\partial u_j} \mid_{\overline{z}}, 2 \leq i \leq n$$

satisfy all the conditions in the approximate variational principle. ■