1 (1.3) Subdifferentials of nonsmooth functions

- X is a Banach space, $\varphi : X \to \overline{\mathbb{R}}$.
- φ is called proper if $\varphi(x) > -\infty \ \forall x \in X$. The domain of φ is defined as

$$\operatorname{dom}\varphi := \{x \in X : \varphi(x) < \infty\} \neq \phi.$$

- The epigraph and hypergraph of φ are defined by

$$\begin{aligned} \operatorname{epi}\varphi \ &: \ = \{(x,a) \in X \times \mathbb{R} : a \geq \varphi(x)\}, \\ \operatorname{hypo}\varphi \ &: \ = \{(x,a) \in X \times \mathbb{R} : a \leq \varphi(x)\}. \end{aligned}$$

- Observe that $\operatorname{gr}\varphi = \operatorname{epi}\varphi \cap \operatorname{hypo}\varphi$.
- $\operatorname{epi}\varphi$ is locally closed at $(\overline{x}, \varphi(\overline{x}))$ if and only if φ is lower semi-continuous around $\overline{x}$.
- $\operatorname{hypo}\varphi$ is locally closed at $(\overline{x}, \varphi(\overline{x}))$ if and only if φ is upper semi-continuous around $\overline{x}$.
- $\operatorname{gr}\varphi$ is locally closed at $(\overline{x}, \varphi(\overline{x}))$ if and only if φ is continuous around $\overline{x}$.
- The notation $x \xrightarrow{\varphi} \overline{x}$ means $(x, \varphi(x)) \to (\overline{x}, \varphi(\overline{x}))$.

1.1 (1.3.1) Basic definitions and results

Proposition 1 *(1.76 basic normals to epigraphs)*
Suppose $\varphi : X \to \overline{\mathbb{R}}$, $(\overline{x}, \overline{\alpha}) \in \operatorname{epi}\varphi$. Then $\lambda \geq 0$ for all $(x^*, -\lambda) \in N((\overline{x}, \overline{\alpha}); \operatorname{epi}\varphi)$.

- There are uniquely defined subsets D and D^∞ of X^* such that

$$N((\overline{x}, \varphi(\overline{x})); \operatorname{epi}\varphi) = \{\lambda(x^*, -1) : x^* \in D, \lambda > 0\} \cup \{(x^*, 0) : x^* \in D^\infty\}.$$

Definition 2 *(1.77 basic and singular subdifferentials)*
Suppose $\varphi : X \to \overline{\mathbb{R}}$, $\overline{x} \in X$, $|\varphi(\overline{x})| < \infty$.

(i) *The set*

$$\partial\varphi(\overline{x}) = \{x^* \in X^* : (x^*, -1) \in N((\overline{x}, \varphi(\overline{x})); \operatorname{epi}\varphi)\}$$

is the basic (limitting) subdifferential of φ at $\overline{x}$. Its elements are the basic subgradients of φ at $\overline{x}$. (We put $\partial\varphi(\overline{x}) = \phi$ if $|\varphi(\overline{x})| = \infty$).

(ii) *The set*

$$\partial^\infty\varphi(\overline{x}) = \{x^* \in X^* : (x^*, 0) \in N((\overline{x}, \varphi(\overline{x})); \operatorname{epi}\varphi)\}$$

is the singular subdifferential of φ at $\overline{x}$. Its elements are singular subdifferentials of φ at $\overline{x}$. (We put $\partial^\infty\varphi(\overline{x}) = \phi$ if $|\varphi(\overline{x})| = \infty$).

Definition 3 *(1.78 upper subgradients)*
Suppose $\varphi : X \to \overline{\mathbb{R}}$, $\overline{x} \in X$, $|\varphi(\overline{x})| < \infty$.

(i) *The set*
$$\partial^{+}\varphi(\overline{x}) = \{x^{*} \in X^{*} : (-x^{*}, 1) \in N((\overline{x}, \varphi(\overline{x})); \mathrm{hypo}\,\varphi)\}$$
is the upper (basic, limitting) subdifferential of φ *at* $\overline{x}$. *(We put* $\partial\varphi(\overline{x}) = \phi$ *if* $|\varphi(\overline{x})| = \infty$*).*

(ii) *The set*
$$\partial^{\infty,+}\varphi(\overline{x}) = \{x^{*} \in X^{*} : (-x^{*}, 0) \in N((\overline{x}, \varphi(\overline{x})); \mathrm{hypo}\,\varphi)\}$$
is the singular upper subdifferential of φ *at* $\overline{x}$. *(We put* $\partial^{\infty}\varphi(\overline{x}) = \phi$ *if* $|\varphi(\overline{x})| = \infty$*).*

- Subgradients and upper subgradients may be substantially different.

Example For $\varphi(x) = -|x|$, $\partial\varphi(0) = \{-1, 1\}$ while $\partial^{+}\varphi(0) = [-1, 1]$.

- $\partial^{+}\varphi(\overline{x}) = -\partial(-\varphi)(\overline{x})$, $\partial^{\infty,+}\varphi(\overline{x}) = -\partial^{\infty}(-\varphi)(\overline{x})$.
- The symmetric and singular symmetric subdifferentials are defined by
$$\begin{aligned} \partial^{0}\varphi(\overline{x}) \quad &: = \partial\varphi(\overline{x}) \cup \partial^{+}\varphi(\overline{x}), \\ \partial^{\infty,0}\varphi(\overline{x}) \quad &= \quad \partial^{\infty,+}\varphi(\overline{x}) \cup \partial^{\infty}\varphi(\overline{x}) \end{aligned}$$
respectively.
- We have
$$\begin{aligned} \partial^{0}(-\varphi)(\overline{x}) &= -\partial^{0}\varphi(\overline{x}), \\ \partial^{\infty,0}(-\varphi)(\overline{x}) &= -\partial^{\infty,0}\varphi(\overline{x}). \end{aligned}$$
- Since properties of upper subdifferentials can be deduced from the corresponding ones for subgradients, we deal from now on only with the subdifferetial ∂.
- For $\Omega \subset X$, the indicator function $\delta(\cdot; \Omega) : X \to \overline{\mathbb{R}}$ is defined by
$$\delta(x; \Omega) = \begin{cases} 0, & x \in \Omega \\ \infty, & x \notin \Omega. \end{cases}$$

Proposition 4 *(1.79 subdifferentials of indicator functions)*
We have
$$\partial\delta(\overline{x}; \Omega) = \partial^{\infty}\delta(\overline{x}; \Omega) = N(\overline{x}; \Omega).$$

- For a function $\varphi : X \to \overline{\mathbb{R}}$, the epigraphical function $E_{\varphi} : X \to \mathbb{R}$ is defined by
$$E_{\varphi}(x) = \{\alpha \in \mathbb{R} : \alpha \geq \varphi(x)\}.$$

- $\operatorname{gr} E_\varphi = \operatorname{epi} \varphi$.
- $D_N^* E_\varphi = D_M^* E_\varphi$ since E_φ takes values in $\mathbb{R}$.
- $\partial \varphi(\overline{x}) = D^* E_\varphi(\overline{x}, \varphi(\overline{x}))(1)$ and $\partial^\infty \varphi(\overline{x}) = D^* E_\varphi(\overline{x}, \varphi(\overline{x}))(0)$.

Theorem 5 *(1.80 subdifferentials from coderivatives of continuous functions)*
Suppose $\varphi : X \to \overline{\mathbb{R}}$ is continuous around $\overline{x} \in X$. Then

$$\begin{aligned} \partial \varphi(\overline{x}) &= D^* \varphi(\overline{x})(1), \\ \partial^\infty \varphi(\overline{x}) &\subset D^* \varphi(\overline{x})(0). \end{aligned}$$

- The inclusion in Theorem 1.80 for ∂^∞ could be strict.

  Example

Corollary 6 *(1.81 subdifferentials of Lipschitzian functions)*
Suppose $\varphi : X \to \overline{\mathbb{R}}$ is Lipschitz continuous around $\overline{x} \in X$ with modulus $\ell \geq 0$. Then

$$\begin{aligned} \partial^\infty \varphi(\overline{x}) &= \{0\}, \\ \|x^*\| &\leq \ell \ \forall x^* \in \partial \varphi(\overline{x}). \end{aligned}$$

- It follows that, if φ is Lipschitz continuous around $\overline{x}$ then $\partial^{\infty,0} \varphi(\overline{x}) = \{0\}$ and $\|x^*\| \leq \ell \ \forall x^* \in \partial^0 \varphi(\overline{x})$.

Corollary 7 *(1.82 subdifferentials of strictly differentiable functions)*
Suppose $\varphi : X \to \overline{\mathbb{R}}$ is strictly differentiable at $\overline{x} \in X$. Then

$$\partial \varphi(\overline{x}) = \partial^+ \varphi(\overline{x}) = \partial^0 \varphi(\overline{x}) = \{\varphi'(\overline{x})\}.$$

- If φ is not strictly differentiable at $\overline{x}$, it could happen that the equalities in Corollary 1.82 do not hold. For example, the function

$$\varphi(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

  is differentiable but not strictly differentiable at 0. Here, $\varphi'(0) = 0$ while $\partial \varphi(\overline{x}) = \partial^+ \varphi(\overline{x}) = [-1, 1]$.