1 (1.2.3) Metric Regularity and Covering

- We use the conventions $d(x,\phi)=\infty$, $\inf\phi=\infty$.

Definition 1 *(1.47 metric regularity)*
$F:X\rightrightarrows Y$, $\operatorname{dom}F\neq\phi$.

(i) *Let $U\subset X,\ V\subset Y$. We say that F is metrically regular on U relative to V if $\exists\,\mu>0$, $\gamma>0$ such that*

$$d\left(x;F^{-1}(y)\right)\leq\mu d\left(y;F(x)\right)\tag{1}$$

for all $x\in U,\ y\in V$ satisfying $d(y;F(x))\leq\gamma$.

(ii) *Let $(\overline{x},\overline{y})\in\operatorname{gr}F$. We say that F is locally metrically regular around $(\overline{x},\overline{y})$ with modulus $\mu>0$ if (i) holds for some nbhds U of x and V of y. The infimum of all moduli μ, denoted $\operatorname{reg}F(\overline{x},\overline{y})$, is called the exact regularity bound of F around $(\overline{x},\overline{y})$.*

(iii) *F is semi-locally metrically regular around $\overline{x}\in\operatorname{dom}F$ (resp. $\overline{y}\in\operatorname{ran}F$) with modulus $\mu>0$ if (i) holds for some nbhds U of x and $V=Y$ (resp. with a nbhd V of $\overline{y}$ and $U=X$).The infimum of all moduli μ is denoted by $\operatorname{reg}F(\overline{x})$ (resp. $\operatorname{reg}F(\overline{y})$).*

Proposition 2 *(1.48 equivalent description of local metric regularity).*
$F:X\rightrightarrows Y$, $\operatorname{dom}F\neq\phi$, $(\overline{x},\overline{y})\in\operatorname{gr}F$, $\mu>0$. *TFAE:*

(a) *F is locally metrically regular around $(\overline{x},\overline{y})$ with modulus μ*

(b) *$\exists$ nbhds U of x and V of y such that (1) holds $\forall x\in U,\ y\in V$.*

(c) *$\exists$ nbhds U of x and V of y such that (1) holds $\forall x\in U,\ y\in V$ for which $F(x)\cap V\neq\phi$.*

Theorem 3 *(1.49 relationships between Lipschitz properties and metric regularity)*
$F:X\rightrightarrows Y$, $\operatorname{dom}F\neq\phi,\ell>0$. $(\overline{x},\overline{y})\in\operatorname{gr}F$

(i) *F is locally Lipschitz-like around $(\overline{x},\overline{y})\in\operatorname{gr}F$ <u>if and only if</u> $F^{-1}:Y\rightrightarrows X$ is locally metrically regular around $(\overline{y},\overline{x})\in\operatorname{gr}F^{-1}$ with the same modulus <u>if and only if</u> $\exists$ nbhds U of x and V of y and $\ell\geq0$ such that*

$$F(x)\cap V\subset F(u)+\ell\left\|u-x\right\|\mathbf{B}_Y\ \forall u\in U,x\in X.$$

In this case,

$$\operatorname{lip}F(\overline{x},\overline{y})=\operatorname{reg}F^{-1}(\overline{y},\overline{x}).$$

(ii) *F is Lipschitzian around $\overline{x}\in\operatorname{dom}F$ if and only if F^{-1} is semi-locally metrically regular around $\overline{x}\in\operatorname{ran}F^{-1}$. In this case*

$$\operatorname{lip}F(\overline{x})=\operatorname{reg}F^{-1}(\overline{y})$$

- Observe that the property of semi-local metric regularity implies that of local metric regularity and one has

$$\begin{aligned}\operatorname{reg} F(\overline{x}) &\geq \sup_{\overline{y}\in F(\overline{x})}\{\operatorname{reg} F(\overline{x},\overline{y})\},\\ \operatorname{reg} F^{-1}(\overline{y}) &\geq \sup_{\overline{x}\in F^{-1}(\overline{y})}\{\operatorname{reg} F(\overline{x},\overline{y})\}.\end{aligned}$$

Proposition 4 *(1.50 relationships between local metric regularity and semi-local metric regularity)*
$F : X \rightrightarrows Y$, $\operatorname{dom} F \neq \phi$.

(i) *Suppose F is closed at $\overline{x} \in \operatorname{dom} F$ and locally compact around $\overline{x}$. Then F is semi-locally metrically regular at $\overline{x}$ if and only if it is locally metrically regular around $(\overline{x}, y)$ for all $y \in F(\overline{x})$. In this case*

$$\operatorname{reg} F(\overline{x}) = \max\{\operatorname{reg} F(\overline{x}, y) : y \in F(\overline{x})\}.$$

(ii) *Suppose F^{-1} is closed at $\overline{y} \in \operatorname{ran} F$ and locally compact around $\overline{y}$. Then F^{-1} is semi-locally metrically regular at $\overline{y}$ if and only if it is locally metrically regular around $(x, \overline{y})$ for all $x \in F^{-1}(\overline{y})$. In this case*

$$\operatorname{reg} F^{-1}(\overline{y}) = \max\left\{\operatorname{reg} F(x, \overline{y}) : x \in F^{-1}(\overline{y})\right\}.$$

Definition 5 *(1.51 covering properties)*
$F : X \rightrightarrows Y$, $\operatorname{dom} F \neq \phi$.

(i) *Let $U \subset X$, $V \subset Y$. We say that F has the covering property on U relative to V if $\exists$ $\kappa > 0$ such that*

$$F(x) \cap V + \kappa r \mathbf{B}_Y \subset F(x + r\mathbf{B}_X), \ \forall r\mathbf{B}_X \subset U, \ r > 0 \tag{2}$$

(ii) *We say that F has the local covering property around $(\overline{x}, \overline{y}) \in \operatorname{gr} F$ with modulus $\kappa > 0$ if (2) holds for some nbhds U of x and V of y. The supremum of all moduli κ, denoted by $\operatorname{cov} F\ (\overline{x}, \overline{y})$ is called the exact covering bound of F around $(\overline{x}, \overline{y})$.*

(iii) *We say that F has the semi-local covering property around $\overline{x} \in \operatorname{dom} F$ with modulus $\kappa > 0$ if (2) holds for some nbhds U of x and $V = Y$. The supremum of all moduli κ is denoted by $\operatorname{cov} F\ (\overline{x})$.*

Theorem 6 *(1.52 relationships between covering and metric regularity)*
$F : X \rightrightarrows Y$, $\operatorname{dom} F \neq \phi$

(i) *F has the semi-local covering property around $\overline{x} \in \operatorname{dom} F$ <u>if and only if</u> F is semi-locally metrically regular around $\overline{x}$.In this case,*

$$\operatorname{cov} F(\overline{x}) = \frac{1}{\operatorname{reg} F(\overline{x})}.$$

(ii) *F has the local covering property around $(\overline{x},\overline{y}) \in \operatorname{gr} F$ if and only if F is locally metrically regular around $(\overline{x},\overline{y})$. In this case*

$$\operatorname{cov} F(\overline{x},\overline{y}) = \frac{1}{\operatorname{reg} F(\overline{x},\overline{y})}.$$

- Relationships for the form

$$0 < \operatorname{cov} F(\overline{x}) = \min\{\operatorname{cov} F(\overline{x},y) : y \in F(\overline{x})\}$$

  do hold for multifunctions that have the semi-local covering property and are closed at $\overline{x}$ and locally compact around that point.

- Relationships for the form

$$\inf\left\{\|x^*\| : x^* \in \widehat{D}^* F(x,y)(y^*)\right\} \geq \rho\|y^*\|$$
$$\forall x \in \overline{x} + \eta B, \quad y \in F(x) \cap \overline{y} + \eta B, \quad y^* \in Y^*$$

  are also associated with multifunctions which have the local metric regularity or the local covering property.

2 (1.2.5) Sequential Normal Compactness of Mappings

Definition 7 *(1.67 sequential normal compactness of Multifunctions)*
$F : X \rightrightarrows Y$, $(\overline{x},\overline{y}) \in \operatorname{gr} F$.

(i) *F is said to be sequentially normally compact at $(\overline{x},\overline{y})$ if $\forall (x_k,y_k) \overset{\operatorname{gr} F}{\to} (\overline{x},\overline{y}), \varepsilon_k \to 0^+, x_k^* \in \widehat{D}^*_{\varepsilon_k} F(x_k,y_k)(y_k^*)$ such that $(x_k^*,y_k^*) \overset{w^*}{\rightharpoonup} (0,0)$ we get $\|(x_k^*,y_k^*)\| \to 0$.*

(ii) *F is said to be partially sequentially normally compact at $(\overline{x},\overline{y})$ if $\forall (x_k,y_k) \overset{\operatorname{gr} F}{\to} (\overline{x},\overline{y}), \varepsilon_k \to 0^+, x_k^* \in \widehat{D}^*_{\varepsilon_k} F(x_k,y_k)(y_k^*)$ such that $y_k^* \to 0$, $x_k^* \overset{w^*}{\rightharpoonup} 0$ we get $\|x_k^*\| \to 0$.*

- F is sequentially normally compact if and only if $\operatorname{gr} F$ is sequentially normally compact.

- Sequential normal compactness implies partial sequential normal compactness.

Proposition 8 *(1.68 partial sequential compactness of Lipschitz-like multifunctions)*
If $F : X \rightrightarrows Y$ is locally Lipschitz-like around $(\overline{x},\overline{y}) \in \operatorname{gr} F$ then F is partially sequentially normally compact.

Corollary 9 *(1.69 partial sequential compactness of single-valued mappings and their inverses)*
If $f : X \to Y$ is Lipschitzian around $\overline{x}$ then

(i) *f is partially sequentially normally compact at $(\overline{x}, f(\overline{x}))$. If $\dim Y < \infty$ then f is sequentially normally compact.*

(ii) *If f is strictly differentiable at $\overline{x}$ and if $f'(\overline{x})$ is surjective then f^{-1} is partially sequentially normally compact around $(f(\overline{x}), \overline{x})$.*

Theorem 10 *(1.70 sequential normal compactness under addition with strictly differentiable mappings)*

If $f : X \to Y$ is strictly differentiable at $\overline{x}$ and $\overline{y} \in F(\overline{x}) + f(\overline{x})$ then $F + f$ is if and only if F is sequentially normally compact (partially sequentially normally compact) at $(\overline{x}, \overline{y} - f(\overline{x}))$.

Theorem 11 *(1.71 sequential normal compactness under composition)*

suppose $G : X \rightrightarrows Y$, $F : Y \rightrightarrows Z$, $\overline{z} \in F \circ G(\overline{x})$, $G(x) \cap F^{-1}(z)$ is inner semi-continuous at $(\overline{x}, \overline{z}, \overline{y})$ and define the function $\Phi : X \times Y \rightrightarrows Z$ by

$$\Phi(x, y) = F(y) + \triangle((x, y); \operatorname{gr} G).$$

Then $F \circ G$ is sequentially normally compact (partially sequentially normally compact) at $(\overline{x}, \overline{z})$ if Φ is sequentially normally compact (partially sequentially normally compact) at $(\overline{x}, \overline{y}, \overline{z})$.