1 Variational Descriptions and Minimality (1.1.5)

Proposition 1 *(variational characterization of ε-normals)*
$\Omega \subset X,\ \overline{x} \in \Omega,\ \varepsilon \geq 0.$
$x^* \in \widehat{N}_\varepsilon(\overline{x};\Omega)$ *if and only if for any* $\gamma > 0$, *the function*

$$\psi(x) = \langle x^*, x - \overline{x} \rangle - (\varepsilon + \gamma)\|x - \overline{x}\|$$

attains a local maximum relative to Ω *at* $\overline{x}$.

Lemma 2 *(smoothing functions in* $\mathbb{R}$*)*
Suppose $\rho : [0,\infty) \to [0,\infty)$ *is such that* $\rho'_+(0)$ *exists,* $\rho(0) = \rho'_+(0) = 0$ *and* $\rho(t) \leq \alpha + \beta t\ \forall t \in [0,\infty),\ \alpha,\beta > 0$. *Then there is a nondecreasing convex* C^1 *function* $\tau : [0,\infty) \to [0,\infty)$ *such that* $\tau(0) = \tau'_+(0) = 0$ *and* $\tau(t) > \rho(t)\ \forall t > 0$.

- A Banach space X is said to admit a Fréchet smooth norm if there is an equivalent norm on X that is Fréchet differentiable at any nonzero point.
- Suppose $\mathcal{S}$ is a given class of functions on a Banach space X. An $\mathcal{S}$-smooth bump function is a function $b : X \to \mathbb{R}$ such that $b \in \mathcal{S}$, $b(x_0) \neq 0$ for some $x_0 \in X$ and $b(x) = 0$ outside some ball in X.
- In what follows we will consider $\mathcal{S}$ to be one of the classes of Fréchet differentiable functions (Fréchet smooth; $\mathcal{S} = \mathcal{F}$), Lipschitzian and Fréchet smooth functions: $\mathcal{S} = \mathcal{LF}$, Lipschitzian and C^1 functions: $\mathcal{S} = \mathcal{L}C^1$.

Theorem 3 *(1.30 variational desc... of...)*
X *a Banach space,* $\phi \neq \Omega \subseteq X$, $\overline{x} \in \Omega$, U *a nbhd of* $\overline{x}$.

(i) *Given* $x^* \in X^*$, *assume* $\exists$ *a function* $S : U \to \mathbb{R}$, *Fréchet differentiable at* $\overline{x}$, $S'(\overline{x}) = x^*$ *and* S *achieves a local maximum relative to* Ω *at* $\overline{x}$. *Then* $x^* \in \widehat{N}(\overline{x};\Omega)$. *Conversely, for every* $x^* \in \widehat{N}(\overline{x};\Omega)$, $\exists$ *a function* $S : X \to \mathbb{R}$, *Fréchet differentiable at* $\overline{x}$, $S'(\overline{x}) = x^*$ *and* S *achieves a local maximum* 0 *relative to* Ω *at* $\overline{x}$.

(ii) *Assume* X *admits a Fréchet smooth renorm. Then for every* $x^* \in \widehat{N}(\overline{x};\Omega)$ *there is a concave Fréchet smooth function* $S : X \to \mathbb{R}$ *that acheives a unique global maximum relative to* Ω *at* $\overline{x}$ *and* $S'(\overline{x}) = x^*$.

(iii) *Assume* X *admits an* $\mathcal{S}$*-smooth bump function, where* $\mathcal{S} \in \{\mathcal{F}, \mathcal{LF}, \mathcal{L}C^1\}$. *Then, for any* $x^* \in \widehat{N}(\overline{x};\Omega)$ $\exists$ *an* $\mathcal{S}$*-smooth function* $S : X \to \mathbb{R}$ *such that (ii) holds.*

Proposition 4 *(1.31: a...)*
$\widehat{\mathcal{N}}$ *a prenormal structure on* X *(i.e.,* $\widehat{\mathcal{N}}(\cdot;\Omega) : X \rightrightarrows X^*$, $x \notin \Omega \Longrightarrow \widehat{\mathcal{N}}(x,\Omega) = \phi$, $\widehat{\mathcal{N}}(x;\Omega) = \widehat{\mathcal{N}}\left(x;\widetilde{\Omega}\right)$ *if* Ω, $\widetilde{\Omega}$ *coincide near* x*).*
Assume:

(M) $\forall x^* \in X^*,\ \varepsilon > 0,\ u \in \Omega \cap B(\overline{x}, \varepsilon)$ *a local minimum of the function*

$$\psi(x) = \langle x^*, x - \overline{x} \rangle + \varepsilon \|x - \overline{x}\|$$

on Ω $\exists$ *a* $v \in u \in \Omega \cap B(\overline{x}, \varepsilon)$ *such that*

$$-x^* \in \widehat{\mathcal{N}}(v; \Omega) + \eta \mathbf{B}_{X^*}\ \forall \eta > \varepsilon.$$

Then

$$N(\overline{x}; \Omega) \subset \mathcal{N}(v; \Omega) := \lim_{x \to \overline{x}} \widehat{\mathcal{N}}(x; \Omega).$$

2 Coderivatives of Set Valued Mappings (1.2)

Here X, Y are Banach spaces, $F : X \rightrightarrows Y$ is a set valued mapping (lower case letters such as f are used to denote single valued mappings).

- F is closed valued (convex valued,...etc.) means that $F(x)$ is closed (convex,...etc.)
- $\operatorname{dom} F := \{x \in X : F(x) \neq \phi\}$,
- $\operatorname{ran} F := \{y \in Y : F^{-1}(y) \neq \phi\}$,
- $\ker F := \{x \in X : 0 \in F(x)\}$,
- $\operatorname{gr} F := \{(x, y) \in X \times Y : y \in F(x)\}$. The norm on $X \times Y$ is taken as $\|(x, y)\| = \|x\| + \|y\|$.
- If $\Omega \subset X,\ F(\Omega) := \{y \in Y : F^{-1}(y) \cap \Omega \neq \phi\}$
- If $\Theta \subset Y,\ F^{-1}(\Theta) := \{x \in X : F(x) \cap \Theta \neq \phi\}$
- $F^{-1} : Y \rightrightarrows X$ is the set valued mapping defined by $F^{-1}(y) := \{x \in X : y \in F(x)\}$
- $\operatorname{dom} F^{-1} = \operatorname{ran} F$, $\operatorname{ran} F^{-1} = \operatorname{dom} F$
- $\operatorname{gr} F^{-1} := \{(y, x) \in Y \times X : (x, y) \in \operatorname{gr} F\} = \{(y, x) \in Y \times X : x \in F^{-1}(y)\}$
- F is positively homogeneous if $0 \in F(0)$ and $F(\alpha x) \supset \alpha F(x)$ $\forall x \in X$ and all $\alpha > 0$.
- F is positively homogeneous $\Longleftrightarrow$ $\operatorname{gr} F$ is a cone in $X \times Y$.
- If F is positively homogeneous we define

$$\|F\| := \sup\{\|y\| : y \in F(x), \|x\| \leq 1\}.$$

2.1 Basic Definitions and Representaions (1.2.1)

Definition 5 *(1.32: coderivatives)*
$F : X \rightrightarrows Y$, $\operatorname{dom} F \neq \phi$.

(i) *The ε-coderivative ($\varepsilon \geq 0$) of F at $(x, y) \in \operatorname{gr} F$ is defined by*

$$\begin{aligned} \widehat{D}^*_\varepsilon F(x,y) &: Y^* \rightrightarrows X^*, \\ \widehat{D}^*_\varepsilon F(x,y)\, y^* &= \left\{ x^* \in X^* : (x^*, -y^*) \in \widehat{N}_\varepsilon((x,y); \operatorname{gr} F) \right\}. \end{aligned}$$

The 0-coderivative is sometimes called the pre-coderivative and is denoted $\widehat{D}^* F(x,y)$.

(ii) *The normal coderivative of F at $(\overline{x}, \overline{y}) \in \operatorname{gr} F$ is defined by*

$$\begin{aligned} D^*_N F(\overline{x},\overline{y}) &: Y^* \rightrightarrows X^*, \\ D^*_N F(\overline{x},\overline{y})\, \overline{y}^* &= w^* - \overline{\lim_{\substack{(x,y)\to(\overline{x},\overline{y}),\\ y^* \overset{w^*}{\to} \overline{y}^*,\\ \varepsilon \to 0^+}}} \widehat{D}^*_\varepsilon F(x,y)\, y^* \\ &= \Big\{ x^* \in X^* : \exists (x_n, y_n) \overset{\operatorname{gr} F}{\to} (\overline{x},\overline{y}), y^*_n \overset{w^*}{\to} \overline{y}^*, \varepsilon_n \to 0^+ \\ &\quad , (x^*_n, -y^*_n) \in \widehat{N}_{\varepsilon_n}((x_n,y_n); \operatorname{gr} F), x^*_n \overset{w^*}{\to} x^*. \Big\} \\ &= \{ x^* \in X^* : (x^*, -y^*) \in N((\overline{x},\overline{y}); \operatorname{gr} F) \}. \end{aligned}$$

If $(\overline{x},\overline{y}) \notin \operatorname{gr} F$ we put $D^*_N F(\overline{x},\overline{y})\, y^* = \phi \ \forall y^* \in Y^*$.

(iii) *The mixed coderivative of F at $(\overline{x}, \overline{y}) \in \operatorname{gr} F$ is defined by*

$$\begin{aligned} D^*_M F(\overline{x},\overline{y}) &: Y^* \rightrightarrows X^*, \\ D^*_M F(\overline{x},\overline{y})\, \overline{y}^* &= w^* - \overline{\lim_{\substack{(x,y)\to(\overline{x},\overline{y}),\\ y^* \to \overline{y}^*,\\ \varepsilon \to 0^+}}} \widehat{D}^*_\varepsilon F(x,y)\, y^* \\ &= \Big\{ x^* \in X^* : \exists (x_n, y_n) \overset{\operatorname{gr} F}{\to} (\overline{x},\overline{y}), y^*_n \to \overline{y}^*, \varepsilon_n \to 0^+ \\ &\quad , (x^*_n, -y^*_n) \in \widehat{N}_{\varepsilon_n}((x_n,y_n); \operatorname{gr} F), x^*_n \overset{w^*}{\to} x^*. \Big\} \end{aligned}$$

Examples 1. Define $F : \mathbb{R} \rightrightarrows \mathbb{R}$ by

$$F(x) = \begin{cases} \left[-\sqrt{1-x^2}, \sqrt{1-x^2}\right], & -1 \leq x \leq 1 \\ \phi & \text{otherwise} \end{cases}.$$

$$\begin{aligned} \widehat{D}^* F(1,0)\, y^* &= \begin{cases} [0, \infty), & y^* = 0 \\ \phi & \text{otherwise} \end{cases}, \\ \widehat{D}^* F(1,1)\, y^* &= \begin{cases} -y^*, & y^* \leq 0 \\ \phi & \text{otherwise} \end{cases}. \end{aligned}$$

2. Define $F : \mathbb{R} \rightrightarrows \mathbb{R}$ by

$$F(x) = \begin{cases} [0,1], & 0 \le x \le 1 \\ \phi & \text{otherwise} \end{cases}.$$

$$\begin{aligned} \widehat{D}^* F(0,0)\, y^* &= \begin{cases} (-\infty, 0], & y^* \ge 0 \\ \phi & \text{otherwise} \end{cases}, \\ \widehat{D}^*_{\varepsilon} F(0,0)\, y^* &= \begin{cases} \phi, & y^* < -\varepsilon \\ \left(-\infty, \sqrt{\varepsilon^2 - y^{*2}}\right], & y^* \in [-\varepsilon, 0] \\ (-\infty, \varepsilon] & \text{otherwise} \end{cases}. \end{aligned}$$

3. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$
4. Define $F : \mathbb{R} \rightrightarrows \mathbb{R}$ by $F(x) = [-|x|, \infty)$
5. Define $F : [0,1] \rightrightarrows [0,1]$ by $F(x) = [0,1]$

- The indicator mapping:

  $\Omega \subset X$, the indicator mapping $\triangle(\cdot;\Omega) : X \to Y$ is defined by

$$\triangle(x;\Omega) = \begin{cases} 0 \in Y \text{ if } x \in \Omega \\ \phi \text{ otherwise} \end{cases}.$$

- $\operatorname{gr} \triangle = \Omega \times \{0\}$.

Proposition 6 *(1.33 coderivative of the indicator mapping)*

$$\begin{aligned} \widehat{D}^*_{\varepsilon} \triangle(\overline{x};\Omega)(y^*) &= \widehat{N}_{\varepsilon} \triangle(\overline{x};\Omega) \;\; \forall y^* \in Y^* \\ D^*_N \triangle(\overline{x};\Omega)(y^*) &= D^*_M \triangle(\overline{x};\Omega)(y^*) = N(\overline{x};\Omega) \;\; \forall y^* \in Y^*. \end{aligned}$$

- Inner semicontinuity:

  $F : X \rightrightarrows Y$ is inner semicontinuous at $\overline{x} \in \operatorname{dom} F$ if

$$\begin{aligned} F(\overline{x}) &= \varliminf_{x \overset{\operatorname{dom} F}{\to} \overline{x}} F(x) \\ &= \left\{ y \in Y : \forall x_k \overset{\operatorname{dom} F}{\to} \overline{x} \; \exists \, y_k \in F(x_k), \; y_k \to y \right\}. \end{aligned}$$

Theorem 7 *(1.34 extremal peroperty of convex valued multifunctions)*

If $F : X \rightrightarrows Y$ is inner semicontinuous at $\overline{x} \in \operatorname{dom} F$ and convex valued around $\overline{x}$ then for any $(\overline{x}, \overline{y}) \in \operatorname{gr} F$ and any $y^* \in \operatorname{dom} D^*_N F(\overline{x};\overline{y})$

$$\langle y^*, \overline{y} \rangle = \min_{y \in F(\overline{x})} \langle y^*, y \rangle.$$

- $\widehat{D}^*F(\overline{x};\overline{y})(y^*) \subset D^*_M F(\overline{x};\overline{y})(y^*) \subset D^*_N F(\overline{x};\overline{y})(y^*)\, \forall y^* \in Y^*$.
- All three functions are positively homogeneous.
- The first inclusion is oftern strict.
- The second inclustion can be strict in infinite dimensional spaces.

Definition 8 *(1.36 graphical regularity of multifunctions)*
$F : X \rightrightarrows Y,\ (\overline{x},\overline{y}) \in \operatorname{gr} F$.

(i) *F is N-regular at $(\overline{x},\overline{y})$ if $D^*_N F(\overline{x},\overline{y}) = \widehat{D}^*F(\overline{x},\overline{y})$.*

(ii) *F is M-regular at $(\overline{x},\overline{y})$ if $D^*_M F(\overline{x},\overline{y}) = \widehat{D}^*F(\overline{x},\overline{y})$.*

- F is N-regular at $(\overline{x},\overline{y})$ if and only if $\operatorname{gr} F$ is normally regular.
- $F : X \rightrightarrows Y$ is convex graph if $\operatorname{gr} F$ is a convex subset of $X \times Y$.

Proposition 9 *(1.37 coderivatives of convex-graph multifunctions)*
Let $F : X \rightrightarrows Y$ be convex graph. Then F is N-regular at $(\overline{x},\overline{y}) \in \operatorname{gr} F$. Furthermore, for any $y^* \in Y^*$

$$\begin{aligned} D^*_M F(\overline{x};\overline{y})(y^*) &= D^*_N F(\overline{x};\overline{y})(y^*) \\ &= \left\{ x^* \in X^* : \langle x^*, \overline{x}\rangle - \langle y^*, \overline{y}\rangle = \max_{(x;y)\in \operatorname{gr} F} \langle x^*, x\rangle - \langle y^*, y\rangle \right\}. \end{aligned}$$

Theorem 10 *(1.38 coderivatives of differentiable mappings)*
Suppose $f : X \to Y$ is Fréchet differentiable at $\overline{x}$. Then

$$\widehat{D}^* f(\overline{x})\, y^* = \{f'(\overline{x})^*\, y^*\}\, \forall y^* \in Y^*.$$

If f is strictly differentiable at $\overline{x}$ then

$$D^*_N f(\overline{x})\, y^* = D^*_M f(\overline{x})\, y^* = \{f'(\overline{x})^*\, y^*\}\, \forall y^* \in Y^*$$

i.e. f is N-regular.

Corollary 11 *(1.39 coderivatives of linear operators)*
If $A \in L(X,Y)$ then A is N-regular and

$$D^*_N A(\overline{x})\, y^* = D^*_M A(\overline{x})\, y^* = \{A^* y^*\}\, \forall y^* \in Y^*, \overline{x} \in X.$$

2.2 Lipschitzian Properties (1.2.2)

In this subsection coderivative conditions are obtained to ensure Lipschitzian properties of mappings.

- For Ω_1, Ω_2 subsets of a metric space Z, let

$$D(\Omega_1, \Omega_2) = \sup_{x \in \Omega_1} d(x, \Omega_2).$$

The Hausdorff distance $d_H(\Omega_1, \Omega_2)$ beteen Ω_1 and Ω_2 is defined to be

$$d_H(\Omega_1, \Omega_2) = \max\{D(\Omega_1, \Omega_2), D(\Omega_2, \Omega_1)\}.$$

- It can be shown that

$$d_H(\Omega_1, \Omega_2) = \inf\{\eta \geq 0 : \Omega_1 \subset \Omega_2 + \eta \mathbf{B}, \, \Omega_2 \subset \Omega_1 + \eta \mathbf{B}\}.$$

- It can be shown that for all $u, v \in Z, \; \Omega \subset Z$

$$\begin{aligned} |d(u, \Omega_1) - d(u, \Omega_2)| &\leq d_H(\Omega_1, \Omega_2), \\ |d(u, \Omega) - d(v, \Omega)| &\leq d(u, v). \end{aligned}$$

Definition 12 *(1.40 Lipschitzian properties of set valued mappings)*
Suppose $F : X \rightrightarrows Y$ and $\operatorname{dom} F \neq \phi$.

(i) *We say that F is Lipschitz-like on $U \subset X$ relative to $V \subset Y$ if U, V are nonempty and there exists an $\ell \geq 0$ such that*

$$F(x) \cap V \subset F(u) + \ell \|x - u\| \mathbf{B}_Y \; \forall x, u \in U \tag{1}$$

(ii) *We say that F is locally Lipschitz-like around $(\overline{x}, \overline{y}) \in \operatorname{gr} F$ with modulus $\ell \geq 0$ if there exist nbhds U of $\overline{x}$ and V of $\overline{y}$ such that (1) holds. The infimum of all $\ell \geq 0$ is called the exact Lipschitzian bound of F around $(\overline{x}, \overline{y})$ and is denoted by $\operatorname{lip} F(\overline{x}, \overline{y})$.*

(iii) *We say that F is Lipschitz continuous on $U \subset X$ if (1) holds with $V = Y$. Furthermore, F is called locally Lipschitzian around $\overline{x}$ with exact bound $\operatorname{lip} F(\overline{x})$ if $V = Y$ in (ii).*

- F is locally Lipschitzian on $U \subset X$ if and only if

$$d_H(F(x), F(y)) \leq \ell \|x - u\| \; \forall x, u \in U,$$

Theorem 13 *(1.41 scalarization of the Lipschitz-like properties)*
Suppose $F : X \rightrightarrows Y$ and $(\overline{x}, \overline{y}) \in \operatorname{gr} F$. TFAE

(a) *F is locally Lipschitz-like around $(\overline{x}, \overline{y})$.*

(b) *The function* $\rho : X \times Y \to \mathbb{R}$ *defined by*

$$\rho(x, y) = d(y, F(x))$$

is locally Lipschitzian around $(\overline{x}, \overline{y})$.

- The property of locally Lipschitzian around $\overline{x} \Longrightarrow$ the property of locally Lipschitzian around $(\overline{x}, \overline{y})$.
- $\operatorname{lip} F(\overline{x}) \geq \sup \{\operatorname{lip} F(\overline{x}, \overline{y}) : \overline{y} \in F(\overline{x})\}$.
- The converse inequality holds in the case given in Theorem 1.42 below.
- $F : X \rightrightarrows Y$ is locally compact around $\overline{x} \in \operatorname{dom} F$ if there exists a nbhd O of $\overline{x}$ and a compact set $C \subset Y$ such that $F(O) \subset C$.
- F is closed (more precisely F is closed graph) at $\overline{x} \in \operatorname{dom} F$ if $\forall y \notin F(\overline{x})$ there exists a nbhd U of $\overline{x}$ and a nbhd V of y such that

$$F(x) \cap V = \phi \ \forall x \in U.$$

Theorem 14 *(1.42 Lipschitz continuity of locally compact mutlifunctions)*
Suppose $F : X \rightrightarrows Y$ *is closed at and locally compact around* $\overline{x} \in \operatorname{dom} F$. *Then* F *is locally Lipschitzian around* $\overline{x}$ *iff it is locally Lipschitz-like around* $(\overline{x}, \overline{y})$ *for all* $\overline{y} \in F(\overline{x})$. *In this case*

$$\operatorname{lip} F(\overline{x}) = \max \{\operatorname{lip} F(\overline{x}, \overline{y}) : \overline{y} \in F(\overline{x})\}.$$

Theorem 15 *(1.43 ε-coderivatives of Lipschitz mappings)*
Suppose $F : X \rightrightarrows Y$, $\overline{x} \in \operatorname{dom} F$ *and* $\varepsilon \geq 0$.

(i) *If* F *is locally Lipschitz-like around* $(\overline{x}, \overline{y}) \in \operatorname{gr} F$ *with modulus* $\ell \geq 0$ *then there exists* $\eta > 0$ *such that*

$$\begin{gathered} \sup \left\{ \|x^*\| : x^* \in \widehat{D}^*_\varepsilon F(x, y) y^* \right\} \leq \ell \|y^*\| + \varepsilon (1 + \ell) \\ \forall x \in B(\overline{x}, \eta), \ y \in F(x) \cap B(\overline{y}, \eta), \ y^* \in Y^*. \end{gathered} \tag{2}$$

Therefore,

$$\operatorname{lip} F(\overline{x}, \overline{y}) \geq \inf_{\eta > 0} \sup \left\{ \left\| \widehat{D}^* F(x, y) \right\| : x \in B(\overline{x}, \eta), \ y \in F(x) \cap B(\overline{y}, \eta) \right\}.$$

(ii) *If* F *is locally Lipschitzian around* $\overline{x}$ *then there exists a* $\eta > 0$ *such that (2) holds* $\forall x \in B(\overline{x}, \eta), \ y \in F(x), \ y^* \in Y^*$. *Therefore,*

$$\operatorname{lip} F(\overline{x}) \geq \inf_{\eta > 0} \sup \left\{ \left\| \widehat{D}^* F(x, y) \right\| : x \in B(\overline{x}, \eta), \ y \in F(x) \right\}.$$

Theorem 16 *(1.44 mixed coderivatives of Lipschitzian mappings)*
Suppose $F : X \rightrightarrows Y$, $\overline{x} \in \operatorname{dom} F$ *and* $\varepsilon \geq 0$.

(i) *If* F *is locally Lipschitz-like around* $(\overline{x}, \overline{y}) \in \operatorname{gr} F$ *then*

$$\|D_M^* F(\overline{x}, \overline{y})\| \leq \operatorname{lip} F(\overline{x}, \overline{y}) < \infty.$$

Therefore,

$$D_M^* F(\overline{x}, \overline{y})(0) = \{0\}.$$

(ii) *If* F *is locally Lipschitzian around* $\overline{x}$ *then*

$$\sup_{\overline{y} \in F(\overline{x})} \|D_M^* F(\overline{x}, \overline{y})\| \leq \operatorname{lip} F(\overline{x}).$$

Therefore,

$$D_M^* F(\overline{x}, \overline{y})(0) = \{0\} \;\; \forall \overline{y} \in F(\overline{x}).$$