1 Generalized Differentiation in Banach Spaces

1.1 Generalized Normals to Nonconvex Sets

1.1.1 Basic Definitions and Some properties

Here X a Banach space, $\phi \neq \Omega \subset X$.

Definition 1 *(Generalized Normals)*

1. $x \in \Omega,\ \varepsilon \geq 0$: *(ε-Normals):*

$$\widehat{N}_{\varepsilon}(x;\Omega) = \left\{ x^* \in X^* : \overline{\lim_{u \xrightarrow{\Omega} x}} \left\langle x^*, \frac{u-x}{\|u-x\|} \right\rangle \leq \varepsilon \right\}.$$

- $\widehat{N}_0(x;\Omega) := \widehat{N}(x;\Omega) :=$ *Fréchet normal or Prenormal (to Ω at x).*
- $x \notin \Omega \overset{def}{\Longrightarrow} \widehat{N}_{\varepsilon}(x;\Omega) = \phi$.

2. $\overline{x} \in \Omega$: *(Normal cone (basic normal)) to Ω at $\overline{x}$:*

$$\begin{aligned} N(\overline{x};\Omega) &= w^*\text{-} \overline{\lim_{\substack{x \xrightarrow{\Omega} \overline{x} \\ \varepsilon \to 0^+}}} \widehat{N}_{\varepsilon}(x;\Omega) \\ &= \left\{ x^* \in X^* : \exists x_n \xrightarrow{\Omega} \overline{x}, \varepsilon_n \to 0^+, x_n^* \in \widehat{N}_{\varepsilon_n}(x_n;\Omega), x_n^* \xrightarrow{w^*} x^* \right\}. \end{aligned}$$

- *Each member of the above set is called a basic/limiting normal.*

Examples (See figures 1 and 2)

- $\widehat{N}_{\varepsilon}(\overline{x};\Omega) = \widehat{N}_{\varepsilon}\left(\overline{x};\overline{\Omega}\right)\ \forall \overline{x} \in \Omega$.
- $\widehat{N}_{\varepsilon}(\overline{x};\Omega) \supset \widehat{N}_{\varepsilon}\left(\overline{x};\widetilde{\Omega}\right)\ \forall \overline{x} \in \Omega \subset \widetilde{\Omega}$.
- $\widehat{N}_{\varepsilon}(\overline{x};\Omega) \subset \widehat{N}_{\widetilde{\varepsilon}}(\overline{x};\Omega)\ \forall 0 \leq \varepsilon \leq \widetilde{\varepsilon}$.
- No monotonicity for $N(\overline{x};\Omega)$. **Examples:** see figures 3 and 4
- $\overline{x} = (\overline{x}_1, \overline{x}_2) \in \Omega = \Omega_1 \times \Omega_2$

$$\begin{aligned} \widehat{N}(\overline{x};\Omega) &= \widehat{N}(\overline{x}_1;\Omega_1) \times \widehat{N}(\overline{x}_2;\Omega_2), \\ N(\overline{x};\Omega) &= N(\overline{x}_1;\Omega_1) \times N(\overline{x}_2;\Omega_2). \end{aligned}$$

  Examples: see figure 5

- $\widehat{N}_{\varepsilon}(\overline{x};\Omega) \supseteq B\left(\widehat{N}(\overline{x};\Omega), \varepsilon\right)$.

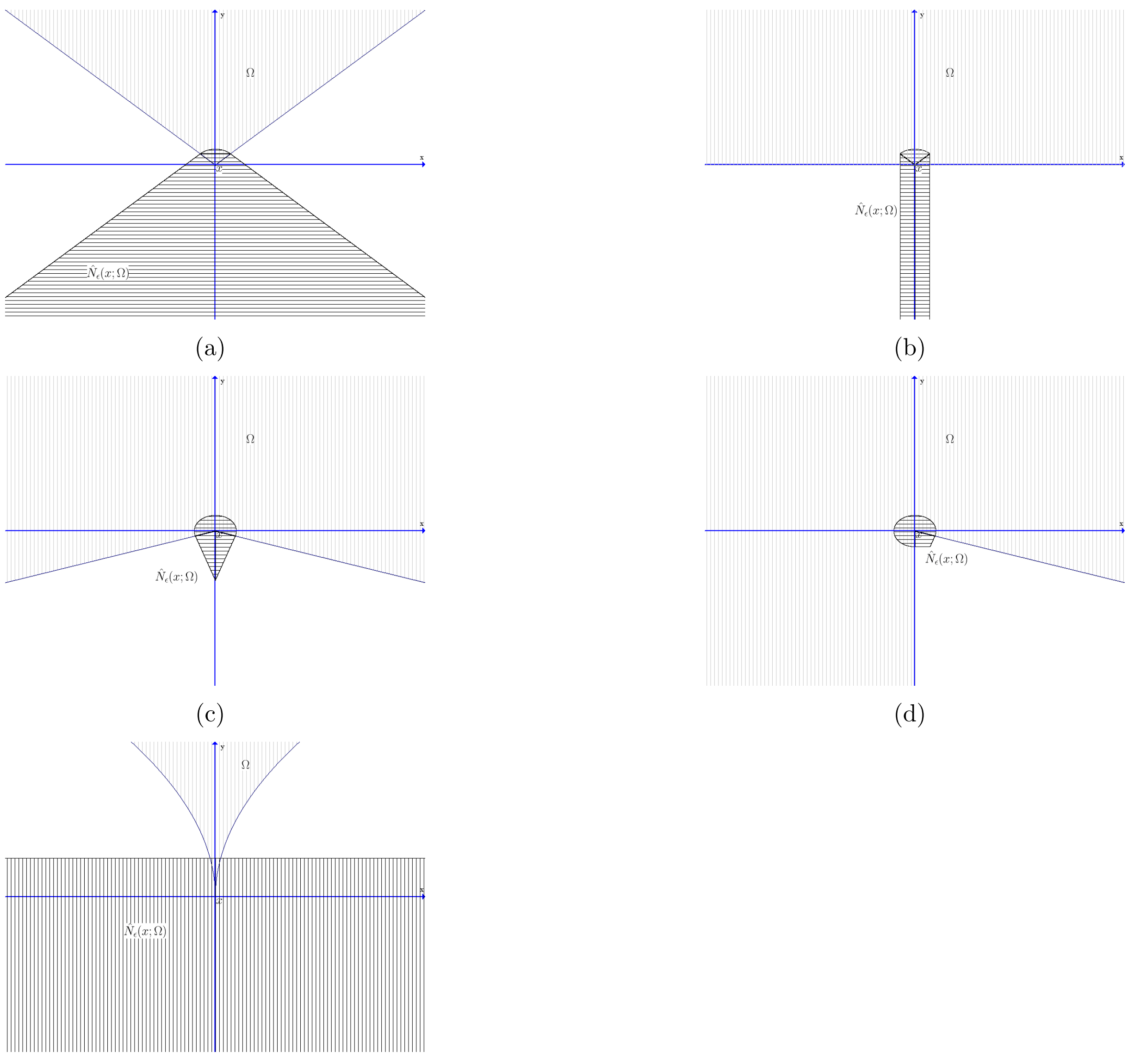


Figure 1: Examples of ϵ-normals

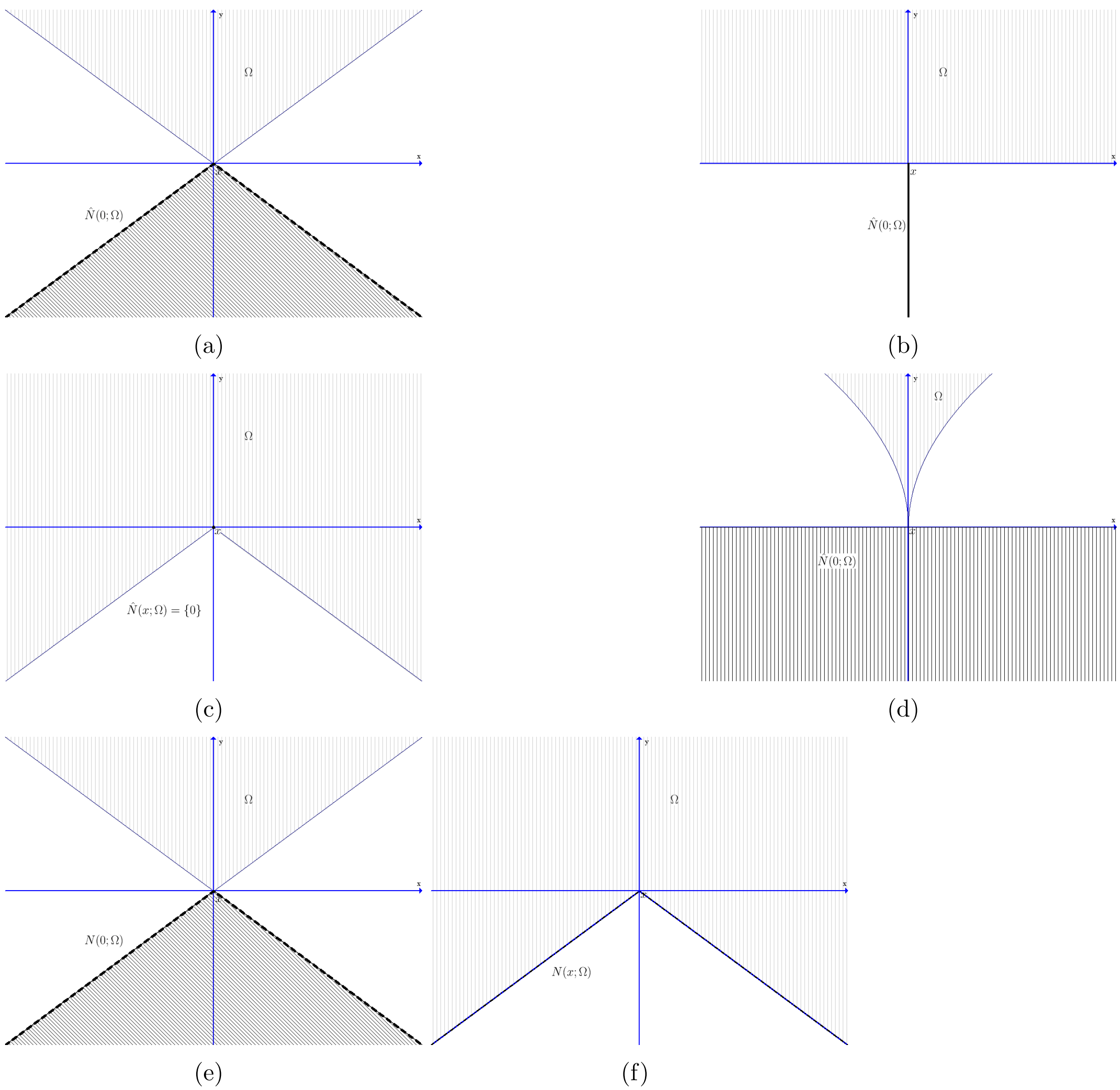
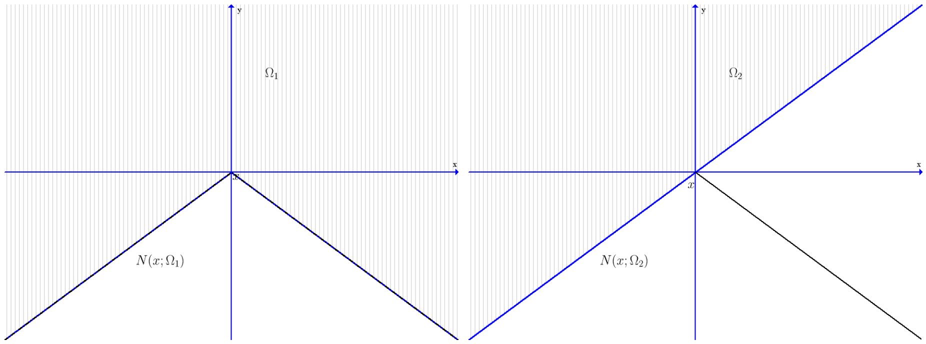
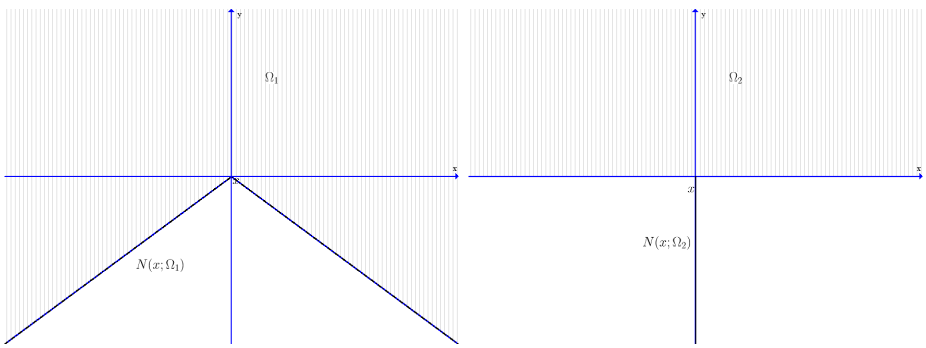


Figure 2: Examples of prenormals and basic normals



(a) (b)

Figure 3: $\Omega_1 \supseteq \Omega_2$, $N(0, \Omega_1) \supseteq N(0, \Omega_2)$



(a) (b)

Figure 4: $\Omega_1 \supseteq \Omega_2$, $N(0, \Omega_1) \cap N(0, \Omega_2) = \{0\}$

- (ε-Normals to convex sets): If Ω is convex, then

$$\widehat{N}_\varepsilon(\overline{x}; \Omega) = \{x^* \in X^* : \langle x^*, x - \overline{x}\rangle \leq \varepsilon \|x - \overline{x}\| \quad \forall x \in \Omega\}.$$

- For $\varepsilon = 0$ we get the classical normal cone of classical convex analysis.

- We always have $N(\overline{x}; \Omega) \supseteq \widehat{N}(\overline{x}; \Omega)$.

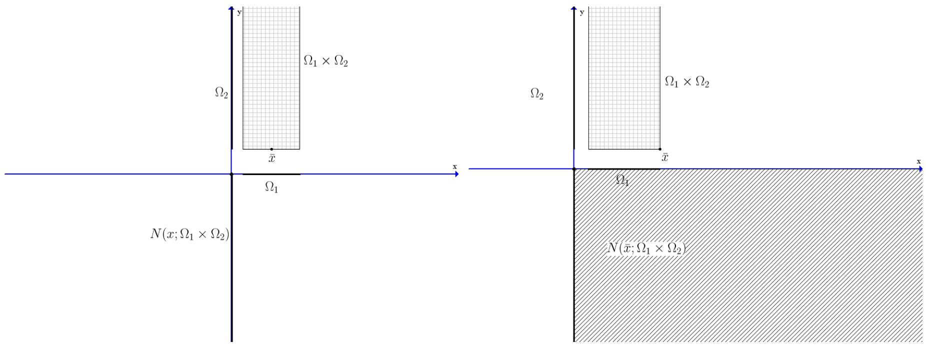
Definition 2 *(normal regularity of sets)*
$\Omega \subset X$ is normally regular at $\overline{x} \in \Omega$ if

$$N(\overline{x}; \Omega) = \widehat{N}(\overline{x}; \Omega).$$

Proposition 3 *Ω locally convex at $\overline{x} \Rightarrow \Omega$ normally regular at $\overline{x}$ and*

$$N(\overline{x}; \Omega) = \{x^* \in X^* : \langle x^*, x - \overline{x}\rangle \leq 0 \ \ \forall x \in \Omega \cap U\},$$

where U is a nbhd of $\overline{x}$.



(a) $\hat{N}(\bar{x}, \Omega_2) = 0 \times \hat{N}(\bar{x}_2, \Omega_2)$ (b) $\hat{N}(\bar{x}, \Omega_2) = \hat{N}(\bar{x}_1, \Omega_1) \times \hat{N}(\bar{x}_2, \Omega_2)$

Figure 5: $\bar{N}(\bar{x}, \Omega_1 \times \Omega_2) = \hat{N}(\bar{x}_1, \Omega_1) \times \hat{N}(\bar{x}_2, \Omega_2)$

- For $\Omega \subset \mathbb{R}^n, x \in \mathbb{R}^n$ the Euclidean projector of x on Ω is defined by

$$\Pi\left(x, \Omega\right) := \left\{u \in \Omega : \|x - u\| = d\left(x, \Omega\right)\right\}.$$

Theorem 4 *(Basic normals in finie dimensions)*
$\Omega \subset \mathbb{R}^n$, $\overline{x} \in \Omega$, Ω is locally closed around $\overline{x}$.

$$\begin{aligned} N\left(\overline{x}; \Omega\right) &= \overline{\lim_{x \to \overline{x}}} \widehat{N}\left(x; \Omega\right) \\ &= \overline{\lim_{x \to \overline{x}}} \left[cone\left(x - \Pi\left(x, \Omega\right)\right)\right]. \end{aligned}$$

Proof. (page 10, lines 8-15)
We need to show that

$$\overline{\lim_{x \to \overline{x}}} \left[\text{cone}\left(x - \Pi\left(x, \Omega\right)\right)\right] \subset \overline{\lim_{x \to \overline{x}}} \widehat{N}\left(x; \Omega\right).$$

To do that, suppose

$$x^* \in \overline{\lim_{x \to \overline{x}}} \left[\text{cone}\left(x - \Pi\left(x, \Omega\right)\right)\right].$$

Then,

$$\exists x_n \xrightarrow{\Omega} \overline{x},\ z_n \in \text{cone}\ \left(x_n - \Pi\left(x_n, \Omega\right)\right), z_n \to x^*.$$

Write

$$z_n = \lambda_n\left(x_n - w_n\right),\ w_n \in \Pi\left(x_n, \Omega\right).$$

Since $\overline{x} \in \Omega$,

$$\|x_n - w_n\| \leq \|x_n - \overline{x}\|.$$

Therefore,

$$x_n - w_n \to 0$$

and

$$w_n \to \overline{x}.$$

We will be done if we show that

$$z_n \in \widehat{N}(w_n;\Omega).$$

Since

$$\begin{aligned} w_n &\in \Pi(x_n,\Omega), \\ 2\langle x_n - w_n, v - w_n\rangle &= -\|x_n - v\|^2 + \|v - w_n\|^2 + \|x_n - w_n\|^2 \\ &\leq \|v - w_n\|^2 \ \ \forall v \in \Omega. \end{aligned}$$

Thus

$$\begin{aligned} \left\langle x_n - w_n, \frac{v - w_n}{\|v - w_n\|}\right\rangle &\leq \frac{1}{2}\|v - w_n\|, \\ \left\langle z_n, \frac{v - w_n}{\|v - w_n\|}\right\rangle &\leq \frac{\lambda_n}{2}\|v - w_n\|. \end{aligned}$$

Therefore,

$$\overline{\lim_{v\to w_n}} \left\langle z_n, \frac{v - w_n}{\|v - w_n\|}\right\rangle \leq 0.$$

■

- The second representation in the above theorem is dependent on the Euclidean norm. **Example.** see figure 6
- The basic normal cone has the following **robustness property** in finite dimensional spaces
$$N(\overline{x};\Omega) = \overline{\lim_{x\to\overline{x}}} N(x;\Omega) \quad \forall\, \overline{x} \in \Omega$$
which is not true in general spaces.
- $N(\overline{x};\Omega)$ is always closed in finite dimensional spaces.

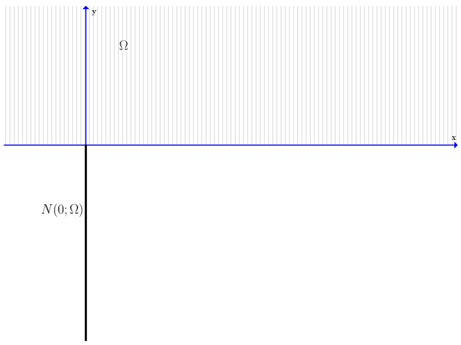
1.1.2 Tangential Approximations

Definition 5 *(Tangent Cones)*
X is a normed space, $\Omega \subset X$, $\overline{x} \in \Omega$.

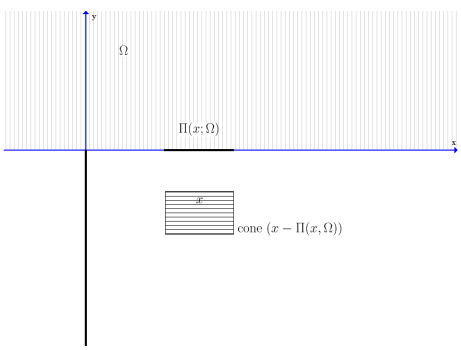
(i) *The Contingent Cone*

$$\begin{aligned} T(\overline{x},\Omega) &= \overline{\lim_{t\to 0^+}} \frac{\Omega - \overline{x}}{t} \\ &= \left\{v \in X : \exists v_n \to v, t_n \to 0^+, \overline{x} + t_n v_n \in \Omega\right\} \\ &= \left\{ v \in X : \forall \varepsilon > 0\ \exists\ \delta > 0\ s.t.\ v \in \bigcup_{t\in(0,\delta)} B\left(\frac{\Omega - \overline{x}}{t}, \varepsilon\right)\right\}. \\ &= \{v \in X : \forall \varepsilon > 0\ \exists\ \delta > 0\ , t \in (0,\delta)\ \ s.t. \overline{x} + t(v + \varepsilon\mathbf{B}) \cap \Omega \neq \phi\}. \end{aligned}$$

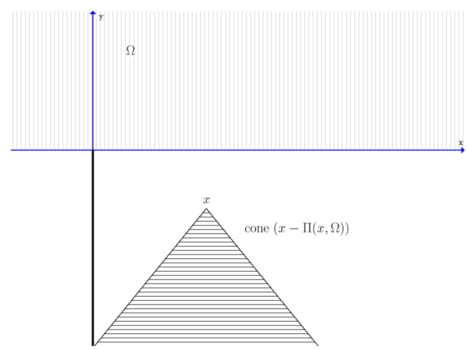
(Observe that $T(\overline{x},\Omega)$ is the limsup of the set valued mapping $F(t) = \frac{\Omega - \overline{x}}{t}$).



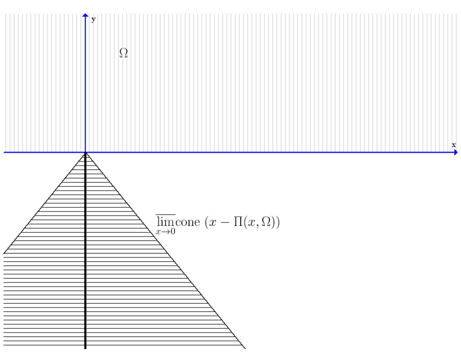
(a) Normals with respect to Euclidean norm



(b)



(c)



(d)

Figure 6:

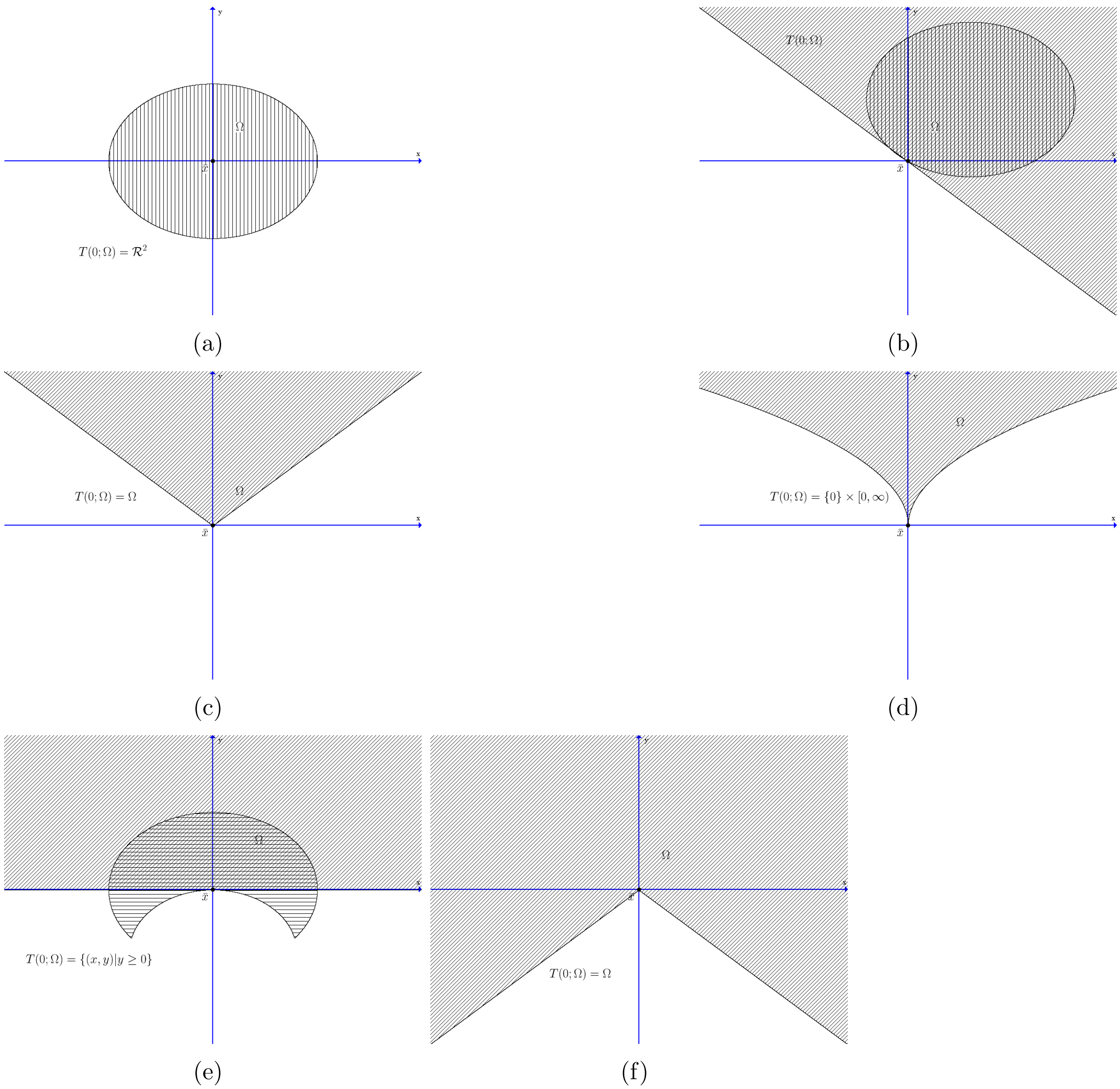


Figure 7: Examples of contingent cones

Examples: *(See figure 7)*

(ii) *The Weak Contingent Cone*

$$T_W(\overline{x},\Omega) = w\text{-}\overline{\lim_{t\to 0^+}} \frac{\Omega - \overline{x}}{t}.$$

(iii) *Clark's Tangent Cone*

$$\begin{aligned}
T_C(\overline{x},\Omega) &= \underset{\substack{x \xrightarrow{\Omega} \overline{x},\\ t\to 0^+}}{\underline{\lim}} \frac{\Omega - \overline{x}}{t} \\
&= \left\{ v \in X : \forall x_n \xrightarrow{\Omega} \overline{x}, t_n \to 0^+ \ \exists v_n \to v,\ \ x_n + t_n v_n \in \Omega \right\} \\
&= \left\{ v \in X : \forall \varepsilon > 0\ \exists\ \delta > 0, \nu > 0\ s.t.\ v \in \bigcap_{\substack{x\in B(\overline{x},\delta)\cap\Omega,\\ t\in(0,\nu)}} B\left(\frac{\Omega - x}{t}, \varepsilon\right) \right\} \\
&= \{ v \in X : \forall \varepsilon > 0\ \exists\ \delta > 0, \nu > 0 \quad s.t.\ \ x + t\,(v + \varepsilon \mathbf{B}) \cap \Omega \neq \phi \forall\, (x,t) \in B\,(\overline{x},\delta)\,, t \in (0,\nu
\end{aligned}$$

- Obviously
$$T_C(\overline{x},\Omega) \subset T(\overline{x},\Omega) \subset T_W(\overline{x},\Omega).$$
- **Examples:** (See figure 8)

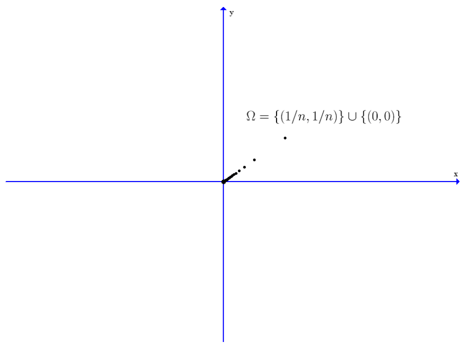
Theorem 6 *(Relationships between tangent cones)*
X a Banach space, $\Omega \subset X$ locally closed around $\overline{x} \in \Omega$.

(i) *We have*
$$\underline{\lim}_{x\xrightarrow{\Omega}\overline{x}} T(x,\Omega) \subset T_C(\overline{x},\Omega) \subset \underline{\lim}_{x\xrightarrow{\Omega}\overline{x}} T_W(x,\Omega).$$

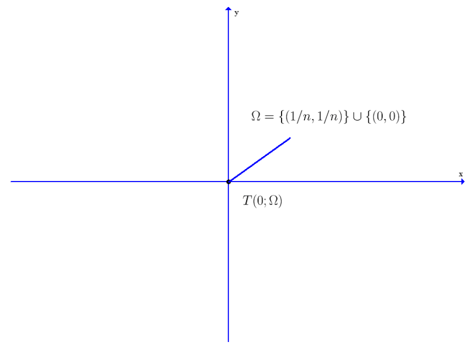
(ii) *If X is reflexive,*
$$T_C(\overline{x},\Omega) = \underline{\lim}_{x\xrightarrow{\Omega}\overline{x}} T_W(x,\Omega).$$

(iii) *If X has a Kadec norm (one in which the weak and norm topologies agree on the boundary of the unit sphere) which is Fréchet differerntiable off the origin then*
$$T_C(\overline{x},\Omega) = \underline{\lim}_{x\xrightarrow{\Omega}\overline{x}} T_W(x,\Omega).$$

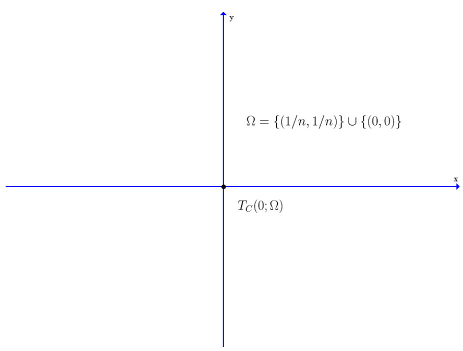
Theorem 7 *(Normal-Tangent relations)*
X a Banach space, $\Omega \subset X$, $\overline{x} \in \Omega$.



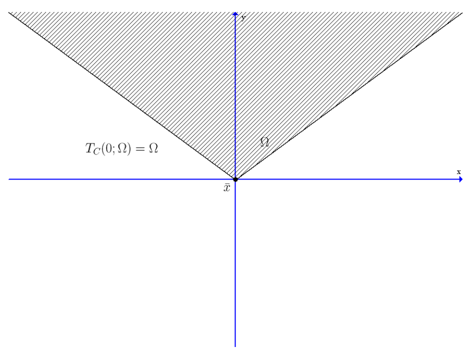
(a)



(b) Contingent cone



(c) Clark's Tangent cone



(d) Clark's Tangent cone

Figure 8: Examples of cones

(i) *For any* $\varepsilon \geq 0$,

$$\widehat{N}_{\varepsilon}(\overline{x};\Omega) \subset \{x^* \in X^* : \langle x^*, v\rangle \leq \varepsilon \|v\| \ \ \forall v \in T(\overline{x},\Omega)\}.$$

Equality holds if $\dim X < \infty$.

(ii) *We have*

$$\widehat{N}(\overline{x};\Omega) \subset \{x^* \in X^* : \langle x^*, v\rangle \leq 0 \ \forall v \in T_W(\overline{x},\Omega)\}.$$

Equality holds if X *is reflexive.*

Denote the dual cone of $T_W(\overline{x},\Omega)$ by $T_W^*(\overline{x},\Omega)$, then:

Corollary 8 *(Normal-Tangent duality)*
X *a reflexive space,* $\Omega \subset X$, $\overline{x} \in \Omega$.

$$\widehat{N}(\overline{x};\Omega) = T_W^*(\overline{x},\Omega) = \{x^* \in X^* : \langle x^*, v\rangle \leq 0 \ \forall v \in T_W(\overline{x},\Omega)\}.$$

When X *is finite dimensional*

$$\widehat{N}(\overline{x};\Omega) = T^*(\overline{x},\Omega).$$

1.1.3 Calculus of Generalized Normals

We begin with some facts about open linear mappings.

Lemma 9 *(a property of open linear operators)*
Suppose $A \in L(X,Y)$ *is such that*

$$A\mu\mathbf{B}_X \supseteq \overline{\mathbf{B}_Y}$$

for some $\rho > 0$. *Given* $x \in X$ *then for any* $y \in Y$, $\exists$ *a* $z \in A^{-1}y$ *such that* $\|x - z\| \leq \rho\|Ax - y\|$. *Consequently,*

$$d\left(x, A^{-1}y\right) \leq \rho\|Ax - y\|.$$

Proof. Fix $x \in X$ and $y \in Y$. The linearity of A gives

$$\begin{aligned}
A\mu r\mathbf{B}_X &\supseteq \overline{r\mathbf{B}_Y} \ \forall r > 0 \\
A(x + \mu r\mathbf{B}_X) &\supseteq Ax + \overline{r\mathbf{B}_Y}. \\
\text{Thus } z &\in Ax + \overline{r\mathbf{B}_Y} \Longrightarrow \exists u \in x + \rho r\mathbf{B}_X \ni \ Au = z, \\
\text{or } \|Ax - z\| &\leq r \Longrightarrow \exists u \in A^{-1}z \ni \|u - x\| < \rho r. \\
\text{In particular } \|Ax - z\| &= r \Longrightarrow \exists u \in A^{-1}y \ni \|u - x\| < \rho r, \\
\text{i.e., } \|Ax - z\| &= r \Longrightarrow \exists u \in A^{-1}y \ni \|u - x\| < \rho\|Ax - z\|.
\end{aligned}$$

The result then follows by setting $r = \|Ax - y\|$ and $z = y$. ■

Theorem 10 *(an open mapping theorem for differentiable functions)*
If f is differentiable at $\overline{x}$ and if $f'(\overline{x})$ is surjective, then there are $\tau, \mu > 0$ such that

$$d\left(\overline{x}, f^{-1}(y)\right) < \mu \|f(\overline{x}) - y\| \;\; \forall y \in f(\overline{x}) + \tau \mathbf{B}_Y.$$

Proof. Let $A = f'(\overline{x})$. Since A is open, there is a $\rho > 0$ such that $A\rho r\mathbf{B}_X \supseteq r\overline{\mathbf{B}_Y} \forall r \leq 1$. Fix $\eta \in (0, \rho^{-1})$. There is a $\delta > 0$ such that

$$\|f(x) - f(\overline{x}) - A(x - \overline{x})\| < \eta \|x - \overline{x}\| \;\; \forall x \in \overline{x} + \delta r \mathbf{B}_X.$$

Therefor,

$$\|f(x) - f(\overline{x}) - A(x - \overline{x})\| < \eta\delta r \; \forall x \in \overline{x} + \delta r \mathbf{B}_X.$$

Thus

$$\begin{array}{rcl} f(\overline{x}) + A(x - \overline{x}) & \in & f(x) + \eta\delta r \mathbf{B}_Y \\ & \subset & f(\overline{x} + \delta r \mathbf{B}_X) + \eta\delta r \mathbf{B}_Y \; \forall x \in \overline{x} + \delta r \mathbf{B}_X. \end{array}$$

Since $x \in \overline{x} + \delta r \mathbf{B}_X \iff (x - \overline{x}) \in \delta r \mathbf{B}_X$, it follows that

$$f(\overline{x}) + A\delta r \mathbf{B}_X \subset f(\overline{x} + \delta r \mathbf{B}_X) + \eta\delta r \mathbf{B}_Y.$$

. Then

$$f(\overline{x}) + \delta r \rho^{-1} \overline{\mathbf{B}_Y} \subset f(\overline{x}) + \delta A r \mathbf{B}_X \subset f(\overline{x} + \delta r \mathbf{B}_X) + \eta\delta r \mathbf{B}_Y.$$

Hence,

$$\begin{array}{rcl} f(\overline{x}) + \delta r \left(\rho^{-1} - \eta\right) \overline{\mathbf{B}_Y} & \subset & f(\overline{x} + \delta \mathbf{B}_X). \\ y & \in & f(\overline{x}) + \delta r \left(\rho^{-1} - \eta\right) \overline{\mathbf{B}_Y} \Longrightarrow \exists x \in \overline{x} + \delta r \mathbf{B}_X \ni f(x) = y \\ \|f(\overline{x}) - y\| & \leq & \delta r \left(\rho^{-1} - \eta\right) \Longrightarrow \exists x \in f^{-1}(y) \ni \|x - \overline{x}\| < \delta r \\ \text{In particular, } \|f(\overline{x}) - y\| & = & \delta r \left(\rho^{-1} - \eta\right) \Longrightarrow \exists x \in f^{-1}(y) \ni \|x - \overline{x}\| < \delta r \\ \text{i.e., } \|f(\overline{x}) - y\| & = & \delta r \left(\rho^{-1} - \eta\right) \Longrightarrow \exists x \in f^{-1}(y) \ni \|x - \overline{x}\| < \left(\rho^{-1} - \eta\right)^{-1} \|f(\overline{x}) - y\|. \end{array}$$

Thus, for any $y \in f(\overline{x}) + \delta(\rho^{-1} - \eta)\mathbf{B}_Y$, $\|f(\overline{x}) - y\| = \delta r(\rho^{-1} - \eta)$ for some $r < 1$ and $\exists x \in f^{-1}(y) \ni \|x - \overline{x}\| < (\rho^{-1} - \eta)^{-1} \|f(\overline{x}) - y\|$. Therefore,

$$d\left(\overline{x}, f^{-1}(y)\right) < \left(\rho^{-1} - \eta\right)^{-1} \|f(\overline{x}) - y\|.$$

The conclusion follows with

$$\tau = \delta\left(\rho^{-1} - \eta\right), \mu = \left(\rho^{-1} - \eta\right)^{-1}.$$

■

Definition 11 *(Strict Differentiability)*

X, Y Banach spaces, $f : X \to Y$.

f is strictly differentiable at $\overline{x}$ if there its Fréchet derivative $f'(\overline{x})$ exists and

$$\lim_{(x,u)\to(\overline{x},\overline{x})} \frac{f(x) - f(u) - f'(\overline{x})(x-u)}{\|x-u\|} = 0.$$

The function

$$r_f(\overline{x};\cdot) : (0,\infty) \to [0,\infty]$$

defined by

$$r_f(\overline{x};\eta) := \sup_{\substack{x,u\in B(\overline{x},\eta),\\ x\neq u}} \frac{\|f(x) - f(u) - f'(\overline{x})(x-u)\|}{\|x-u\|}$$

is called the rate of strict differentiability.

- If f is strictly differentiable at $\overline{x}$ then $r_f(\overline{x};\eta) \to 0$ as $\eta \to 0^+$.

Example

$$f(x) = \begin{cases} x^2, & x \text{ rational} \\ 0, & \text{Otherwise} \end{cases}.$$

$f'(0) = 0,$ f is not strictly differentiable. To see this, take $x = \frac{1}{n},\ u = \frac{1}{n} + \sqrt{2}\frac{1}{n^2}$.

$$\frac{f(x) - f(u) - f'(0)(x-u)}{|x-u|} = \frac{1}{\sqrt{2}} \nrightarrow 0.$$

- $f \in C^1$at $\overline{x} \implies f$ is strictly differetiable at $\overline{x}$ but the converse is not always true.

Example A function which is strictly differentiable at 0 and nowhere else:

$$f(x) = \begin{cases} x^4, & x \text{ rational} \\ x^2, & \text{Otherwise} \end{cases}.$$

- f strictly differentiable at $\overline{x} \implies f$ is locally Lipschitz around $\overline{x}$: There exists a $l > 0$ and a nbhd U of $\overline{x}$ such that

$$\|f(x) - f(u)\| \le l\,\|x-u\| \ \ \forall x, u \in U.$$

Lemma 12 *(Open mapping for strictly differentiable functions)*

If f is strictly differentiable at $\overline{x}$ and if $\overline{y} = f(\overline{x})$ then there exist $\tau, \mu, \eta > 0$ such that for all $y \in \overline{y} + \tau\mathbf{B}_Y$ and $x \in \overline{x} + \eta\mathbf{B}_X$,

$$d\left(x, f^{-1}(y)\right) \le \mu\,\|f(x) - y\|.$$

Proof. Following the proof of the similar lemma for differentiable functions we conclude that there exist $\rho, \sigma, \eta > 0$ such that for all $0 < r \leq 1$,

$$\begin{aligned} f(x) + \rho r \overline{\mathbf{B}_Y} &\subseteq f(x + \sigma r \mathbf{B}_X) \, \forall x \in \overline{x} + \eta \mathbf{B}_X \\ \text{ie., } \|f(x) - y\| &\leq \rho r \Longrightarrow \exists v \in f^{-1}(y) \ni \|x - v\| < \sigma r \\ \text{in particular, } \|f(x) - y\| &= \rho r \Longrightarrow \exists v \in f^{-1}(y) \ni \|x - v\| < \sigma \rho^{-1} \|f(x) - y\| . \end{aligned}$$

We may assume that $\eta \leq \frac{\rho}{2l}$. Thus, for $y \in \overline{y} + \frac{\rho}{2}\mathbf{B}_Y$ and $x \in \overline{x} + \eta \mathbf{B}_X$,

$$\begin{aligned} \|f(x) - y\| &< \|f(x) - \overline{y}\| + \frac{\rho}{2} \\ &\leq l \|x - \overline{x}\| + \frac{\rho}{2} \\ &\leq \eta l + \frac{\rho}{2} \leq \rho. \end{aligned}$$

Therefore,

$$\|f(x) - y\| = \rho r$$

for some $r < 1$. Hence, there exists a $v \in f^{-1}(y) \ni \|x - v\| < \sigma \rho^{-1} \|f(x) - y\|$. It follows that

$$d\left(x, f^{-1}(y)\right) \leq \sigma \rho^{-1} \|f(x) - y\|$$

The result follows with $\mu = \sigma \rho^{-1}$ and $\tau = \frac{\rho}{2}$. ■

Theorem 13 *(ε-normals to inverse images under differentiable mappings)*
X, Y Banach spaces, $f : X \to Y$, $\Theta \subset Y$, $\overline{y} = f(\overline{x}) \in \Theta$.

(i) *If f is Fréchet differentiable at $\overline{x}$ then there is a $c_1 > 0$ such that*

$$\widehat{N}_\varepsilon\left(\overline{x}; f^{-1}(\Theta)\right) \supset f'(\overline{x})^* \widehat{N}_{c_1 \varepsilon}(\overline{y}; \Theta) \ \forall \varepsilon \geq 0.$$

(ii) *If f is strictly differentiable at $\overline{x}$ and $f'(\overline{x})$ is surjective then there is a $c_2 > 0$ such that*

$$\widehat{N}_\varepsilon\left(\overline{x}; f^{-1}(\Theta)\right) \subset B\left(f'(\overline{x})^* \widehat{N}_{c_2 \varepsilon}(\overline{y}; \Theta), \varepsilon\right). \tag{1}$$

(iii) *If f is continuous around $\overline{x}$ and $f'(\overline{x})$ exists and is surjective and if* $\dim Y < \infty$ *then (1) holds.*

Proof. The proof of (ii) follows the following steps.

Step1: Given $x^* \in \widehat{N}_\varepsilon(\overline{x}; f^{-1}(\Theta))$, show, using the strict differentiability of f that $|\langle x^*, x\rangle| \leq \varepsilon \|x\| \ \forall x \in \ker A$, where $A = f'(\overline{x})$.

Step2: Show that there is a $\widehat{x}^* \in (\ker A)^\perp$ such that $\|x^* - \widehat{x}^*\| \leq \varepsilon$.

Step3: Since $R(A)$ is closed, $(\ker A)^\perp = \left(R(A^*)^\top\right)^\perp = \overline{R(A^*)}^{w^*} = R(A^*)$ (see Rudin p 96,101).

Step4: It follows that $\widehat{x}^* = A^* y^*$ for some $y^* \in Y^*$. Show that $y^* \in \widehat{N}_{c_2\varepsilon}(\overline{y};\Theta)$ for some $c_2 > 0$.

The details are as follows.

Step1: Put $A = f'(\overline{x})$ and let $x^* \in \widehat{N}_\varepsilon(\overline{x}; f^{-1}(\Theta))$. Suppose $x \in \ker A$. Then

$$\|f(\overline{x}+tx) - f(\overline{x})\| = o(t) \text{ as } t \longrightarrow 0^+.$$

It follows from Lemma 12 that, for t sufficiently small, there exists $x_t \in f^{-1}(f(\overline{x}))$ such that $\|x_t - (\overline{x}+tx)\| \le \mu \|f(\overline{x}+tx) - f(\overline{x})\|$. Hence,

$$\|x_t - (\overline{x}+tx)\| = o(t) \text{ as } t \longrightarrow 0^+.$$

This means that

$$\begin{aligned} \frac{x_t - (\overline{x}+tx)}{t} &\rightarrow 0 \text{ as } t \longrightarrow 0^+ \\ \frac{x_t - \overline{x}}{t} &\rightarrow x \text{ as } t \longrightarrow 0^+ \end{aligned}$$

Hence,

$$\left\langle x^*, \frac{x}{\|x\|} \right\rangle = \lim_{t \longrightarrow 0^+} \left\langle x^*, \frac{x_t - \overline{x}}{\|x_t - \overline{x}\|} \right\rangle \le \overline{\lim_{x \to \overline{x}}} \left\langle x^*, \frac{x - \overline{x}}{\|x - \overline{x}\|} \right\rangle \le \varepsilon.$$

Since $-x \in \ker A$, $|\langle x^*, x\rangle| \le \varepsilon \|x\|$.

Step2: It follows from Step1 that

$$\|x^*|_{\ker A}\| \le \varepsilon.$$

By the Hahn-Banach theorem, we extend $x^*|_{\ker A}$ to a functional $\widetilde{x}^* \in X^*$ such that $\|\widetilde{x}^*\| \le \varepsilon$. Let $\widehat{x}^* = x^* - \widetilde{x}^*$. Then for any $x \in \ker A$, $\langle \widehat{x}^*, x\rangle = 0$ and $\widehat{x}^* \in (\ker A)^\perp$. Also $\|x^* - \widehat{x}^*\| = \|\widetilde{x}^*\| \le \varepsilon$.

Step3: Since $R(A)$ is closed, $R(A^*)$ is weak star closed. The rest of the statement is self explanatory.

Step4: It follows from Step3 that $\widehat{x}^* = A^* y^*$ for some $y^* \in Y^*$. Now by Lemma 12 again, there are $\tau, \mu, \eta > 0$ such that for each $y \in \overline{y} + \tau \mathbf{B}_Y \cap \Theta$ there is an $x \in f^{-1}(y)$ such that $\|x - \overline{x}\| \le \mu \|f(x) - \overline{y}\| = \mu \|y - \overline{y}\|$. For $y \to \overline{y}$ we choose $x \to \overline{x}$ as indicated. Then

$$\begin{aligned} \left\langle y^*, \frac{y - \overline{y}}{\|y - \overline{y}\|} \right\rangle &= \left\langle y^*, \frac{f(x) - f(\overline{x})}{\|x - \overline{x}\|} \right\rangle \frac{\|x - \overline{x}\|}{\|y - \overline{y}\|} \\ &\le \mu \left(\left\langle y^*, \frac{A(x - \overline{x})}{\|x - \overline{x}\|} \right\rangle + \left\langle y^*, \frac{f(x) - f(\overline{x}) - A(x - \overline{x})}{\|x - \overline{x}\|} \right\rangle \right) \\ &= \mu \left(\left\langle \widehat{x}^*, \frac{x - \overline{x}}{\|x - \overline{x}\|} \right\rangle + \left\langle y^*, \frac{f(x) - f(\overline{x}) - A(x - \overline{x})}{\|x - \overline{x}\|} \right\rangle \right) \\ &\le \mu \left(\left\langle x^*, \frac{x - \overline{x}}{\|x - \overline{x}\|} \right\rangle + \|x^* - \widehat{x}^*\| + \left\langle y^*, \frac{f(x) - f(\overline{x}) - A(x - \overline{x})}{\|x - \overline{x}\|} \right\rangle \right) \end{aligned}$$

Therefore,

$$\begin{aligned}\overline{\lim_{\substack{y\to\overline{y}\\ y\in\Theta}}} &\leq \mu\overline{\lim_{\substack{x\to\overline{x}\\ x\in f^{-1}(\Theta)}}}\left\langle x^{*},\frac{x-\overline{x}}{\|x-\overline{x}\|}\right\rangle+\mu\varepsilon\\ &\leq 2\mu\varepsilon.\end{aligned}$$

Thus, if we choose $c_2 = 2\mu$, we get

$$\widehat{N}_{c_2\varepsilon}\left(\overline{y};\Theta\right).$$

■

Corollary 14 *(prenormals to inverse images under differentiable mappings)*

$$\widehat{N}\left(\overline{x};f^{-1}\left(\Theta\right)\right)\supset f'\left(x\right)^{*}\widehat{N}\left(\overline{y};\Theta\right).$$

Equality holds if $f'\left(\overline{x}\right)$ *is surjective and either* $\dim Y<\infty$ *or* f *is strictly differentiable at* $\overline{x}$.

Theorem 15 *(basic normals to inverse images under strictly differentiable mappings)*
If f *is strictly differentiable at* $\overline{x}$ *and if* $f'\left(\overline{x}\right)$ *is surjective then*

$$N\left(\overline{x};f^{-1}\left(\Theta\right)\right)=f'\left(x\right)^{*}N\left(\overline{y};\Theta\right).$$

Theorem 16 *(normal regularity of inverse images under strictly differentiable mappings)*
If f *is strictly differentiable at* $\overline{x}$ *and* $f'\left(\overline{x}\right)$ *is surjective then* $f^{-1}\left(\Theta\right)$ *is normally regular at* $\overline{x}$ *if and only if* Θ *is normally regular at* $\overline{y}=f\left(\overline{x}\right)$.

1.1.4 Sequential Normal Compactness of Sets

Definition 17 *(Sequential normal Compactness SNC)*
A subset $\Omega\subset X$ *is SNC at* $\overline{x}\in\Omega$ *if for any sequences* $x_n\overset{\Omega}{\to}\overline{x}$, $\varepsilon_n\to 0^{+}$ *and any* $x_n^{*}\in\widehat{N}_{\varepsilon_n}\left(x_n;\Omega\right)$ *such that* $x_n^{*}\overset{w^{*}}{\to}0$, *one has* $\|x_n^{*}\|\to 0$.

- If Ω is SNC then $\overline{\Omega}$ is SNC.
- The affine hull of fa set Ω is defined by

$$\operatorname{aff}\left(\Omega\right)=\left\{\sum_{i=1}^{n}\alpha_i x_i:\sum_{i=1}^{n}\alpha_i=1,\alpha_i\in\mathbb{R},x_i\in\Omega,1\leq i\leq n,\ n\in\mathbb{N}\right\}.$$

- For any $x\in\overline{\operatorname{aff}}\left(\Omega\right)$, $\overline{\operatorname{aff}}\left(\Omega\right)-x$ is a closed linear subspace of X that does not depend on x.

- We define the codimension of the affine hull as

$$\text{codim } \overline{\text{aff}}(\Omega) = \dim X / \left(\overline{\text{aff}}(\Omega) - x \right)$$

  where $x \in \Omega$.

- The relative interior $\text{ri}\,\Omega$ of a set Ω is defined to be the interior of Ω with respect to $\overline{\text{aff}}(\Omega)$.

Theorem 18 *(finite codimension of SNC sets)*
$\Omega \subset X$ *SNC at* $\overline{x} \in \Omega \Longrightarrow \text{codim } \overline{\text{aff}}(\Omega \cap U) < \infty$ *for any nbhd* U *of* $\overline{x}$*. Moreover,* $\text{ri}\,\Omega \neq \phi \Longrightarrow \Omega$ *is SNC at every* $\overline{x} \in \Omega$ *iff* $\text{codim } \overline{\text{aff}}(\Omega) < \infty$.

Theorem 19 *(SNC for inverse images under strictly differentiable mappings)*
If f *is strictly differentiable at* $\overline{x}$ *and if* $f'(x)$ *is surjective then* $f^{-1}(\Theta)$ *is SNC at* $\overline{x} \in f^{-1}(\Theta)$ *iff* Θ *is SNC at* $\overline{y} = f(\overline{x})$.

Theorem 20 *(SNC for inverse images under linear operators)*
If $A \in L(X,Y)$ *has closed range, and if* $\Theta \subset AX$ *is SNC at* $\overline{y} = A\overline{x} \in \Theta$ *then* $A^{-1}(\Theta)$ *is SNC at* $\overline{x}$.

- In what follows, sufficient conditions for SNC are given.

Definition 21 *(epi-Lipschitzian and compactly epi-Lipschitzian sets)*
$\Omega \subset X,\ \overline{x} \in \Omega$.

(i) Ω *is compactly epi-Lipschitzian (CEL) around* $\overline{x}$ *if there is a compact set* $C \subset X$*, a nbhd* U *of* $\overline{x}$*, a nbhd* O *of* $0 \in X$ *and a* $\gamma > 0$ *such that*

$$\Omega \cap U + tO \subset \Omega + tC \ \forall t \in (0, \gamma). \tag{2}$$

(ii) Ω *is epi-Lipschitzian (EL) around* $\overline{x}$ *if* C *in (2) can be chosen as a singleton.*

Proposition 22 *(EL property of convex sets)*
If $\Omega \subset X$ *is convex then* Ω *is EL around any* $\overline{x} \in \Omega$ *if and only if* $\text{int}\,\Omega \neq \phi$.

Theorem 23 *(SNC of CEL sets)*
If $\Omega \subset X$ *is CEL around* $\overline{x} \in \Omega$ *then* Ω *is SNC at* $\overline{x}$.