Extremal Principle in Variational Analysis

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1 (2.3.3) Smooth Variational Principles

- Condition (c) of Theorem 2.26 can be rewritten as

$$s(x) := \varphi(\overline{x}) - \varepsilon/\lambda d(x, \overline{x}) \leq \varphi(x) \ \ \forall x \in X.$$

  Then we may think of s as supporting φ from below. s is usually called a supporting function.

- If s belongs to a smoothness class $\mathcal{S}$, we has that the $\mathcal{S}$-variational principle holds in X.

- An $\mathcal{S}$-variational principle is called concave if $\mathcal{S}$ consists of concave functions.

- The smoothness classes to be considered here are $\mathcal{F}$ (for Fréchet differentiable functions), $\mathcal{LF}$ (for Lipschitzian and Fréchet differentiable functions) and $\mathcal{LC}^1$ (for Lipschitzian and continuously differentiable functions). Thus we will be considering

$$\mathcal{S} \in \left\{\mathcal{F}, \mathcal{LF}, \mathcal{LC}^1\right\}.$$

Theorem 1 *(2.31: smooth variational principles in Asplund spaces)*
Let X be a Banach space and $\mathcal{A}$ be the class of proper lsc functions $\varphi : X \to \overline{\mathbb{R}}$ bounded from below. Given $\varepsilon > 0, \lambda > 0,$ we have:

(i) *If X admits a Fréchet smooth renorm then for every $\varphi \in \mathcal{A}$ and $x_0 \in X$ with $\varphi(x_0) < \inf_X \varphi + \varepsilon$ there exist $\overline{x} \in X$ and a concave Fréchet differentiable function $s : X \to \mathbb{R}$ such that*

$$\begin{aligned} \|\overline{x} - x_0\| &< \lambda, \quad \varphi(\overline{x}) \leq \inf_X \varphi + \varepsilon, \\ \varphi(\overline{x}) &= s(\overline{x}), \qquad \varphi(x) \geq s(x) + \|x - \overline{x}\|^2 \ \ \forall x \in X \end{aligned} \tag{1}$$

and

$$\|s'(\overline{x})\| \leq \varepsilon/\lambda. \tag{2}$$

(ii) *Let X admit an $\mathcal{S}$-smooth bump function. Then for every $\varphi \in \mathcal{A}$ and $x_0 \in X$ with $\varphi(x_0) < \inf_X \varphi + \varepsilon$ there exist $\overline{x} \in X$ satisfying (1), an $\mathcal{S}$-smooth bump function $b : X \to \mathbb{R}$ and a constant $c \in \mathbb{R}$ such that*

$$\varphi(\overline{x}) = b(\overline{x}) + c, \qquad \varphi(x) \geq b(x) + c \ \forall x \in X$$

and

$$\|b'(\overline{x})\| \leq \varepsilon/\lambda.$$

Moreover, one can find $\mathcal{S}$-smooth functions $s : X \to \mathbb{R}$ and $\theta : X \to [0, \infty)$ such that $\theta(x) = 0$ only if $x = 0$, $\theta(x) \leq \|x\|^2$ for $x \in \mathbf{B}$,

$$\varphi(\overline{x}) = s(\overline{x}), \qquad \varphi(x) \geq s(x) + \theta(x - \overline{x}) \ \ \forall x \in X$$

and (2) is satisfied.

(iii) *Conversely, the concave $\mathcal{F}$-smooth variational principle holds in X if and only if X admits a Fréchet smooth renorm, and the $\mathcal{S}$-smooth variational principle holds in X if and only if X admits an $\mathcal{S}$-smooth bump function.*

2 (2.4) Representations and Characterizations in Asplund Spaces

2.1 (2.4.1) Subgradients, Normals and Coderivatives in Asplund Spaces

- $\mathcal{SL}(\overline{x})$ denotes the class of functions (φ_1, φ_2) of proper functions $\varphi_i : X \to \overline{\mathbb{R}}$ such that φ_1 is Lipschitzian around $\overline{x}$ and φ_2 is lsc around $\overline{x}$. ($\mathcal{SL}$ for semi-Lipschitzian).

Lemma 2 *(2.32: subgradient description of the extremal principle)*
Let X be a Banach space. Then

(i) *Let the approximate extremal principle hold in $X \times \mathbb{R}$ for any extremal system of two closed sets. Assume that $(\varphi_1, \varphi_2) \in \mathcal{SL}(\overline{x})$ and $\varphi_1 + \varphi_2$ attains a local minimum at $\overline{x}$. Then for any $\eta > 0$ there are $x_i \in \overline{x} + \eta\mathbf{B}$ with $|\varphi_i(x_i) - \varphi_i(\overline{x})| \leq \eta,\ i = 1, 2$ such that*

$$0 \in \widehat{\partial}\varphi_1(x_1) + \widehat{\partial}\varphi_2(x_2) + \eta\mathbf{B}^*. \tag{3}$$

(ii) *Conversely, suppose that for any $(\varphi_1, \varphi_2) \in \mathcal{SL}(\overline{x})$ such that $\overline{x} \in X$ is a local minimum of $\varphi_1 + \varphi_2$ and any $\eta > 0$ there exist $x_i \in \overline{x} + \eta\mathbf{B}$ such that $|\varphi_i(x_i) - \varphi_i(\overline{x})| \leq \eta,\ i = 1, 2$ and (3) is satisfied. Then the approximate extremal principle holds for every extremal system of two closed sets in X.*

Proof. To show (i) assume that $(\varphi_1,\varphi_2)\in\mathcal{SL}(\overline{x})$ and (wlog) that $\overline{x}=0$, $\varphi_1(0)=\varphi_2(0)=0$, that φ_1 is Lipschitzian on $\eta\mathbf{B}$ with modulus $\ell>0$ and that φ_2 is lsc on $\eta\mathbf{B}$ for some fixed $\eta>0$. Defined the sets (closed around $(0,0)$)

$$\Omega_1=\operatorname{epi}\varphi_1,\quad \Omega_2:=\operatorname{hypo}-\varphi_2.$$

We can easily check that the system $\{\Omega_1,\Omega_2;(0,0)\}$ is an extremal system in $X\times\mathbb{R}$. Applying the approximate extremal priciple we find, for each $\varepsilon>0$ $(x_i,\alpha_i)\in\Omega_i$ and $(x_i^*,\lambda_i)\in X^*\times\mathbb{R}$, $i=1,2$ such that

$$(x_1^*,-\lambda_1)\in\widehat{N}\left((x_1,\alpha_1);\Omega_1\right),\qquad(-x_2^*,\lambda_2)\in\widehat{N}\left((x_2,\alpha_2);\Omega_2\right)$$

and

$$\begin{cases}\|(x_i,\alpha_i)\|<\varepsilon,\\ \frac{1}{2}-\varepsilon<\|(x_i^*,\lambda_i)\|<\frac{1}{2}+\varepsilon,\quad i=1,2,\\ \|(x_1^*,-\lambda_1)+(-x_2^*,\lambda_2)\|<\varepsilon.\end{cases}$$

Since $(x_1^*,\lambda_1)\neq0$ for sufficiently somall ε, we get $(x_1,\alpha_1)\in\operatorname{bd}\Omega_1$, which gives $\alpha_1=\varphi_1(x_1)$ and $\lambda_1>0$. (see the proof of Theorem 2.28). Thus, $\frac{x_1^*}{\lambda_1}\in\widehat{\partial}\varphi_1(x_1)$ and $\|x_1^*\|\leq\ell\lambda_1$. It follows from $\|(x_1^*,\lambda_1)\|>\frac{1}{2}-\varepsilon$ that

$$\lambda_1>\frac{1-2\varepsilon}{2(\ell+1)}>\varepsilon$$

for sufficiently small ε. Since $\lambda_2-\lambda_1<-\varepsilon$, we get

$$\lambda_2>\lambda_1-\varepsilon>\frac{1-2\varepsilon}{2(\ell+1)}-\varepsilon>\sqrt{\varepsilon}$$

for sufficiently small ε. This in turn gives (see the proof of Theorem 2.28) that $\alpha_2=\varphi(x_2)$. Therefore, $-\frac{x_2^*}{\lambda_2}\in\widehat{\partial}\varphi_2(x_2)$. Furthermore,

$$\begin{aligned}\left\|\frac{x_1^*}{\lambda_1}-\frac{x_2^*}{\lambda_2}\right\| &= \frac{\|\lambda_2x_1^*-\lambda_1x_2^*\|}{\lambda_1\lambda_2}=\frac{\|\lambda_1(x_1^*-x_2^*)+x_1^*(\lambda_2-\lambda_1)\|}{\lambda_1\lambda_2}\\ &\leq \frac{\|x_1^*-x_2^*\|}{\lambda_2}+\frac{\|x_1^*\|}{\lambda_1\lambda_2}|\lambda_2-\lambda_1|\\ &\leq \frac{\varepsilon}{\lambda_2}+\frac{\varepsilon\ell}{\lambda_2}=\frac{\varepsilon}{\lambda_2}(1+\ell)\leq\sqrt{\varepsilon}(1+\ell)\leq\eta\end{aligned}$$

for sufficiently small ε. Therefore, $0\in\widehat{\partial}\varphi_1(x_1)+\widehat{\partial}\varphi_2(x_2)+\eta\mathbf{B}^*$. Finally, $\|x_i\|<\varepsilon\leq\eta$ for sufficiently small ε, $i=1,2$. This completes the proof of (i).

To prove (ii), let $\{\Omega_1,\Omega_2,\overline{x}\}$ be an extremal system in X. Let $\varepsilon>0$ be given and choose $a\in\varepsilon^2/2\mathbf{B}$ such that

$$(\Omega_1+a)\cap\Omega_2=\phi.$$

Define the function $\varphi:X\times X\to\mathbb{R}$ by

$$\varphi(u,v)=\frac{1}{2}\|u-v+a\|,\ (u,v)\in X\times X.$$

Observe that $\varphi(\overline{x},\overline{x}) < (\varepsilon/2)^2$ and that $\varphi(u,v) > 0$ on $\Omega_1 \times \Omega_2$.

Applying Ekland's variational principle to the function φ on $\Omega_1 \times \Omega_2$ we get the existence of a $(\overline{u},\overline{v}) \in \Omega_1 \times \Omega_2$ such that $\|(\overline{u},\overline{v}) - (\overline{x},\overline{x})\| < \varepsilon/2$ and

$$\varphi(\overline{u},\overline{v}) \leq \varphi(u,v) + \varepsilon/2 \|(u,v) - (\overline{u},\overline{v})\| \;\; \forall (u,v) \in \Omega_1 \times \Omega_2.$$

The function

$$\begin{aligned} \psi(u,v) \quad &: \;= \varphi(u,v) + \varepsilon/2 \|(u,v) - (\overline{u},\overline{v})\| + \delta((u,v);\Omega_1 \times \Omega_2) \\ &= \;\; \varphi_1(u,v) + \varphi_2(u,v), \end{aligned}$$

where

$$\varphi_1(u,v) := \varphi(u,v) + \varepsilon/2 \|(u,v) - (\overline{u},\overline{v})\|$$

and

$$\varphi_2(u,v) := \delta((u,v);\Omega_1 \times \Omega_2)$$

acheives its minimum at $(\overline{u},\overline{v})$. Since $(\varphi_1,\varphi_2) \in \mathcal{SL}(\overline{x})$, we have from the hypothesis in (ii), that there exist $(u_i,v_i) \in (\overline{u},\overline{v}) + \varepsilon/2(\mathbf{B} \times \mathbf{B})$ with $|\varphi_i(u_i,v_i) - \varphi_i(\overline{u},\overline{v})| \leq \varepsilon/2,\ i = 1,2$ and

$$\begin{aligned} (0,0) \;\; &\in \;\; \widehat{\partial}\varphi_1(u_1,v_1) + \widehat{\partial}\varphi_2(u_2,v_2) + \varepsilon/2(\mathbf{B}^* \times \mathbf{B}^*) \\ &= \;\; \widehat{\partial}\left(\frac{1}{2}\|u - v + a\| + \varepsilon/2 \|(u,v) - (\overline{u},\overline{v})\|\right)\Big|_{(u_1,v_1)} \\ &\quad\;\; + \widehat{N}((u_2,v_2);\Omega_1 \times \Omega_2) + \varepsilon/2(\mathbf{B}^* \times \mathbf{B}^*) \\ &= \;\; \frac{1}{2}\widehat{\partial}\|u - v + a\|\Big|_{(u_1,v_1)} + \varepsilon/2(\mathbf{B}^* \times \mathbf{B}^*) \\ &\quad\;\; + \widehat{N}((u_2,v_2);\Omega_1 \times \Omega_2) + \varepsilon/2(\mathbf{B}^* \times \mathbf{B}^*) \\ &= \;\; \frac{1}{2}\widehat{\partial}\|u - v + a\|\Big|_{(u_1,v_1)} + \widehat{N}((u_2,v_2);\Omega_1 \times \Omega_2) + \varepsilon(\mathbf{B}^* \times \mathbf{B}^*). \end{aligned}$$

Thus,

$$(0,0) = \frac{1}{2}(u_1^*, -u_2^*) + (y_1^*, y_2^*),$$

where

$$\begin{aligned} (y_1^*, y_2^*) \;\; &\in \;\; \widehat{N}((u_2,v_2);\Omega_1 \times \Omega_2) + \varepsilon(\mathbf{B}^* \times \mathbf{B}^*), \\ (u_1^*, -u_2^*) \;\; &\in \;\; \widehat{\partial}\|u - v + a\|\Big|_{(u_1,v_1)}. \end{aligned}$$

At this point we get stuck. It appears that this proof cannot be fixed without extra assumptions on the norm of the space X. ■

Theorem 3 *(2.34 subdifferential representations in Asplund Spaces)*

Let X be a Banach space, $\overline{x} \in X$ and $\mathcal{A}(\overline{x})$ be the class of proper functions $\varphi : X \to \overline{\mathbb{R}}$ lsc around $\overline{x} \in \operatorname{dom}\varphi$. TFAE.

(a) *X is Asplund.*

(b) *For every* $\overline{x} \in X$ *and every* $\varphi \in \mathcal{A}(\overline{x})$ *one has*

$$\partial \varphi(\overline{x}) = \varlimsup_{x \xrightarrow{\varphi} \overline{x}} \widehat{\partial} \varphi(x).$$

(c) *For every* $\overline{x} \in X$ *and every* $\varphi \in \mathcal{A}(\overline{x})$ *and every* $\varepsilon > 0$ *one has*

$$\partial_\varepsilon \varphi(\overline{x}) = \widehat{\partial} \varphi(x) + \varepsilon \mathbf{B}^*.$$

Theorem 4 *(2.35 basic normals in Asplund Spaces)*
Let X be a Banach space. TFAE.

(a) *X is Asplund.*

(b) *For every closed set* $\Omega \subset X$ *and every* $\overline{x} \in \Omega$,

$$N(\overline{x}; \Omega) = \varlimsup_{x \to \overline{x}} \widehat{N}(x; \Omega).$$

Corollary 5 *(2.36 coderivatives of mappings between Asplund Spaces)*
Let $F : X \rightrightarrows Y$ *be a multifunction between the Asplund spaces X and Y. Assume* $\operatorname{gr} F$ *is closed around* $(\overline{x}, \overline{y})$. *Then*

$$D_N^* F(\overline{x}, \overline{y})(\overline{y}^*) = \varlimsup_{\substack{(x,y) \to (\overline{x}, \overline{y}), \\ y^* \xrightarrow{w^*} \overline{y}^*}} \widehat{D}^* F(x, y)(y^*), \ \forall \overline{y}^* \in Y^*,$$

and

$$D_M^* F(\overline{x}, \overline{y})(\overline{y}^*) = \varlimsup_{\substack{(x,y) \to (\overline{x}, \overline{y}), \\ y^* \to \overline{y}^*}} \widehat{D}^* F(x, y)(y^*), \ \forall \overline{y}^* \in Y^*.$$

2.2 (2.4.2) Representations of Singular Subgradients and Horizontal Normals to Graphs and Epigraphs

Lemma 6 *(2.37 horizontal Fréchet normals to epigraphs)*
Let X be Asplund and let $\varphi : X \to \overline{\mathbb{R}}$ *be a proper function lsc around* $\overline{x} \in \operatorname{dom} \varphi$. *Then for every* $x^* \in X^*$ *with* $(x^*, 0) \in \widehat{N}((\overline{x}, \varphi(\overline{x})); \operatorname{epi} \varphi)$ *there are sequences* $x_k \xrightarrow{\varphi} \overline{x}$, $\lambda_k \searrow 0$ *and* $x_k^* \in \lambda_k \widehat{\partial} \varphi(x_k)$ *such that* $\|x_k^* - x^*\| \to 0$ *as* $k \to \infty$.

Theorem 7 *(2.38 singular subgradients in Asplund spaces)*
Let X be an Asplund space. Assume that $\varphi : X \to \overline{\mathbb{R}}$ *be a proper function lsc around* $\overline{x} \in \operatorname{dom} \varphi$. *Then*

$$\partial^\infty \varphi(\overline{x}) = \varlimsup_{\substack{x \xrightarrow{\varphi} \overline{x}, \\ \lambda \searrow 0}} \lambda \widehat{\partial} \varphi(x) = \varlimsup_{\substack{x \xrightarrow{\varphi} \overline{x}, \\ \varepsilon, \lambda \searrow 0}} \lambda \widehat{\partial}_\varepsilon \varphi(x).$$

Corollary 8 *(2.39 subdifferential description of sequential normal epicompactness)*

Let X be an Asplund space. Assume that $\varphi : X \to \overline{\mathbb{R}}$ be a proper function lsc around $\overline{x} \in \operatorname{dom}\varphi$. Then φ is SNEC if and only if for any sequences $x_k \xrightarrow{\varphi} \overline{x}$, $\lambda_k \searrow 0$ and $x_k^* \in \lambda_k \widehat{\partial}\varphi(x_k)$ one has

$$\left[x_k^* \xrightarrow{w^*} 0\right] \Longrightarrow \|x_k^*\| \to 0.$$

Theorem 9 *(2.40 horizontal normals to graphs of continuous functions)*

Let X be an Asplund space. Assume that $\varphi : X \to \overline{\mathbb{R}}$ be finite and continuous around some point $\overline{x} \in X$. Then:

(i) *If $(x^*, 0) \in \widehat{N}((\overline{x}, \varphi(\overline{x})); \operatorname{epi}\varphi)$ then there exist sequences $x_k \xrightarrow{\varphi} \overline{x}$, $\lambda_k \searrow 0$ and $x_k^* \to x^*$ such that*

$$x_k^* \in \widehat{\partial}(\lambda_k\varphi)(x_k) \cup \widehat{\partial}(-\lambda_k\varphi)(x_k) \ \ \forall k.$$

(ii) $D^*\varphi(\overline{x})(0) = \partial^\infty\varphi(\overline{x}) \cup \partial^\infty(-\varphi)(\overline{x})$.