Basic Differential Geometry

March 20, 2007

1 Curves in $\mathbb{R}^2$

Consider a function C from an interval $J = [a, b]$ into $\mathbb{R}^2$ with derivative $C_p(p) \neq 0 \; \forall p \in J$. One can think of the point $\mathbf{x} = C(p)$ as traversing a curve γ while p traverses J. We do not call C itself a curve. Instead, any function $\widetilde{C}$ obtained from C by a suitable change of the parameter is regarded as representing the same curve γ as C.

The derivative C_p of C has the geometric interpretation as a tangent vector. To see this, suppose p_0 is a point of J and that p is sufficiently close to p_0 that $p \in J$. Define the linear function $y(p)$ by

$$y(p) = C(p_0) + (p - p_0) C_p(p_0).$$

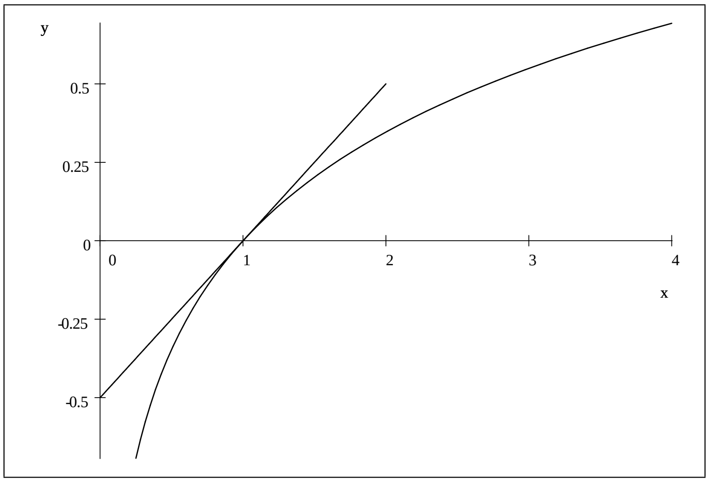
This line passes through $C(p_0)$ and has direction $C_p(p_0)$. Then

$$\frac{C(p) - y(p)}{p - p_0} = \frac{C(p) - C(p_0)}{p - p_0} - C_p(p_0) \to 0 \text{ as } p \to p_0.$$

Hence $y(p)$ is tangent to the function $C(p)$ at p_0 (recall that the functions f and g are tangent at x_0 if $\lim_{x \to x_0} \frac{|f(x)-g(x)|}{|x-x_0|} = 0$).

Example Let $C(p) = \begin{pmatrix} p^2 \\ \log p \end{pmatrix}$, $J = [1/2, 2]$. Find the tangent line at $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. In this example $p_0 = 1. C_p(p) = \begin{pmatrix} 2p \\ 1/p \end{pmatrix}$, $C_p(1) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. The tangent line is$y(p) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (p-1)\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2p-1 \\ p-1 \end{pmatrix}$ in parametric form. That is $x(p) = 2p - 1$ and $y(p) = p - 1$. In Cartesian form, by eliminating p, we get

$$2y = x - 1.$$



Observe also that the curve is traversed from the point $\begin{pmatrix} 1/4 \\ -\log 2 \end{pmatrix}$ to the point $\begin{pmatrix} 4 \\ \log 2 \end{pmatrix}$.

C is a simple curve: it does not cross itself.

Definition 1 *Let C be any parametric representation of a curve γ on an interval $J = [a, b]$. Let $\phi : J \to [\alpha, \beta]$ be any continuously differentiable function such that*

1. $\phi'(p) > 0 \; \forall \; p \in J$,
2. $\phi(a) = \alpha, \phi(b) = \beta$.

Write $q = \phi(p)$. Then the function $\widetilde{C} : [\alpha, \beta] \to \mathbb{R}^2$ defined by

$$\begin{aligned} \widetilde{C}(q) &= \widetilde{C}(\phi(p)) \\ &= C(p) \end{aligned}$$

is called a reparametrizaion of the curve γ.

The derivatives also satisfy the following relations

$$\begin{aligned} \widetilde{C}_q(q) &= \frac{d}{dq} C(p) \\ &= \frac{d}{dp} C(p) \frac{dp}{dq} \\ &= \frac{1}{\phi'(p)} C_p(p). \end{aligned}$$

If $\widetilde{C}$ and C are parametrizations of γ according to Definition 1, we will say that $\widetilde{C}$ and C are equivalent. The curve γ will be identified with any of its equivalent representations. γ will be called a simple arc if it does not cross itself, i.e, if for some parametrization C, we have $C(p_1) \neq C(p_2)$ for $p_1 \neq p_2$. If the aforementioned condition is satisfied except when $p_1 = a, p_2 = b$ (where $C(a) = C(b)$), we say that γ is a simple closed curve.

The tangent vector $\frac{dC}{dp}$ will be denoted by T. That is

$$\begin{aligned} T &= \frac{dC}{dp} = C_p \\ &= \begin{pmatrix} x_p \\ y_p \end{pmatrix}. \end{aligned}$$

The assumption that $C_p(p) \neq 0$ for all $p \in J$ implies that $|T| = |C_p| > 0$. Thus the curve γ has a well defined tangent vector for all $p \in J$.

1.1 Parametrization by arclength

The arc length on γ is defined by

$$s = \int_a^p |C_p(\sigma)| \, d\sigma.$$

Lemma 2 *The arc length is independent of the parametrization of* γ.

Proof. Suppose C and $\widetilde{C}$ are parametrizations of γ according to Definition 1. Then

$$\begin{aligned} C &= \widetilde{C} \circ \phi, \\ C_p &= \widetilde{C}_q \circ \phi \phi'. \end{aligned}$$

In terms of the parametrization $\widetilde{C}$, the arclength is given by

$$\int_\alpha^q \left| \widetilde{C}_q(\sigma) \right| d\sigma.$$

Using the change of variable

$$\sigma = \phi(\tau), \; d\sigma = \phi'(\tau) \, d\tau$$

we get

$$\begin{aligned} \int_\alpha^q \left| \widetilde{C}_q(\sigma) \right| d\sigma &= \int_a^p |C_p(\tau)| \frac{1}{\phi'(\tau)} \phi'(\tau) \, d\tau \\ &= \int_a^p |C_p(\tau)| \, d\tau. \end{aligned}$$

■

Let us set $s = \phi(p)$, where $\phi(p) = \int_a^p |C_p(\tau)| \, d\tau$. Then $\phi : J \to [0, l(\gamma)]$, where $l(\gamma)$ is the length of the curve γ, $\phi'(p) = |C_p(\tau)| > 0$. It follows that $\widetilde{C}(s) = \widetilde{C}(\phi(p))$ is a reparametrization of γ. Furthermore, we have

$$\begin{aligned} \widetilde{C}_s &= C_p \frac{dp}{ds} = \frac{C_p}{|C_p|} \qquad (1) \\ &= \overrightarrow{T}, \end{aligned}$$

where $\overrightarrow{T}$ is the unit tangent vector to the curve γ at the point $C(p)$. This says that the parametrization by arclength is such that the tangent vectors always have unit length.

For the next lemma we need to introduce two vector products in $\mathbb{R}^2$: the cross product (or outer product) and the dot product (or inner product).

Definition 3 *Let* $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ *and* $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ *be two vectors in* $\mathbb{R}^2$. *The cross (or outer) product of* x *and* y *is the scalar*

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = x_1 y_2 - x_2 y_1.$$

The dot (or inner) product is the scalar

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2.$$

Observe that $|\mathbf{x} \times \mathbf{y}|$ is the area of the parallelogram whose sides are the vectors $\mathbf{x}, \mathbf{y}$.

We have the following change of variable formula.

Lemma 4 *Let* γ *be a curve with parametrizations* $C, \widetilde{C}$ *according to Definition 1, then*

$$\widetilde{C}_q \times \widetilde{C}_{qq} = \frac{1}{\phi'^3} \left(C_p \times C_{pp} \right).$$

Proof. Writing $C = \widetilde{C} \circ \phi$ and using the chain rule, we obtain

$$\begin{aligned} C_p &= \widetilde{C}_q \circ \phi \phi', \\ C_{pp} &= \widetilde{C}_{qq} \circ \phi \phi'^2 + \widetilde{C}_q \circ \phi \phi''. \end{aligned}$$

Thus, observing that $\phi' > 0$ and $\widetilde{C}_q \times \widetilde{C}_q = 0$,

$$\begin{aligned} C_p \times C_{pp} &= \phi' \widetilde{C}_q \times \left(\phi'^2 \widetilde{C}_{qq} + \phi'' \widetilde{C}_q \right) \\ &= \phi'^3 \widetilde{C}_q \times \widetilde{C}_{qq}. \end{aligned}$$

■

Lemma 5 *If* $\widetilde{C}$ *is a parametrization of* γ *by arclength, then*

$$\widetilde{C}_{ss} = \frac{C_p \times C_{pp}}{|C_p|^3} \overrightarrow{N}, \tag{2}$$

where $\overrightarrow{N}$ *is the unit normal vector obtained by rotating* $\overrightarrow{T}$ *through an angle of* $\frac{\pi}{2}$ *CCW.*

Proof. Differentiating (1) once again with respect to s, we get

$$\begin{aligned}
\widetilde{C}_{ss} &= \frac{d}{dp}\frac{C_p}{|C_p|}\frac{dp}{ds} \\
&= \frac{1}{|C_p|}\frac{|C_p|\,C_{pp} - \frac{\langle C_p, C_{pp}\rangle}{|C_p|}C_p}{|C_p|^2} \\
&= \frac{C_{pp} - \frac{\langle C_p, C_{pp}\rangle}{|C_p|^2}C_p}{|C_p|^2} \\
&= \frac{C_{pp} - \left\langle \overrightarrow{T}, C_{pp}\right\rangle \overrightarrow{T}}{|C_p|^2}.
\end{aligned}$$

Now, we observe that $C_{pp} - \left\langle \overrightarrow{T}, C_{pp}\right\rangle \overrightarrow{T}$ is the component of C_{pp} in the direction normal to $\overrightarrow{T}$. This component can also written as $|C_{pp}|\sin\nu$, where ν is the angle from $\overrightarrow{T}$ to C_{pp}. Observe that the sign of ν agrees with the sign of $\overrightarrow{T} \times C_{pp}$ (by the right hand rule for cross products) and therefore $\sin\nu = \frac{\overrightarrow{T}\times C_{pp}}{|C_{pp}|}$. Therefore,

$$C_{pp} - \left\langle \overrightarrow{T}, C_{pp}\right\rangle \overrightarrow{T} = \left(\overrightarrow{T} \times C_{pp}\right)\overrightarrow{N}.$$

We get (2) upon writing $\overrightarrow{T}$ back as $\frac{C_p}{|C_p|}$. ■

Since C_s is in the direction of $\overrightarrow{T}$ and C_{ss} is in the direction of $\overrightarrow{N}$, we always have

$$\left\langle \widetilde{C}_s, \widetilde{C}_{ss}\right\rangle = 0.$$

This can also obtained by differentiating the equation

$$\left|\widetilde{C}_s\right|^2 = \left\langle \widetilde{C}_s, \widetilde{C}_s\right\rangle = 1$$

with respect to s.

1.2 Rotations

In Lemma 5 we talked about rotating the unit vector $\overrightarrow{T}$ through an angle of $\frac{\pi}{2}$ CCW. This rotation can be written algebraically as

$$\overrightarrow{N} = R\left(\frac{\pi}{2}\right)\overrightarrow{T},$$

where $R(\tau)$ is the rotation operator through angle τ

$$R(\tau) = \begin{bmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{bmatrix}.$$

A rotation operator has the following important properties:

Lemma 6 *Assume $R(\tau)$ is the rotation operator through angle τ then*

1. $R(\tau)^* = R(\tau)^{-1} = R(-\tau)$; *here $R(\tau)^*$ is the transpose of $R(\tau)$.*
2. $R(\tau)$ *preserves the inner product.*
3. $R(\tau)$ *preserves norms.*

As a consequence, $R(\tau)$ preserves the angle between two vectors

Proof. The first property is an immediate consequence of the definition of $R(\tau)$. To prove the second property, assume that $\mathbf{x}, \mathbf{y}$ are two vectors in $\mathbb{R}^2$. Then

$$\begin{aligned} \langle R(\tau)\mathbf{x}, R(\tau)\mathbf{y}\rangle &= \langle R(\tau)^* R(\tau)\mathbf{x}, \mathbf{y}\rangle \\ &= \left\langle R(\tau)^{-1} R(\tau)\mathbf{x}, \mathbf{y}\right\rangle \\ &= \langle \mathbf{x}, \mathbf{y}\rangle . \end{aligned}$$

To see the third property, assume $\mathbf{x}$ is a vector in $\mathbb{R}^2$. Then

$$\begin{aligned} |R(\tau)\mathbf{x}|^2 &= \langle R(\tau)\mathbf{x}, R(\tau)\mathbf{x}\rangle \\ &= \langle \mathbf{x}, \mathbf{x}\rangle = |\mathbf{x}|^2 . \end{aligned}$$

■

1.3 The Curvature

The geometric curvature κ at a point $\mathbf{x}_0$ on γ is defined as the reciprocal of the radius of the osculating circle at $\mathbf{x}_0$. This is found by passing a circle through $\mathbf{x}_0$ and two adjacent points on γ, computing the radius of the circle and then taking the limit as the two adjacent points tend to $\mathbf{x}_0$. This is equivalent to the formula

$$\rho = \frac{ds}{d\theta}, \kappa = \frac{1}{\rho},$$

where θ is the angle between the tangent $\overrightarrow{T}$ and the positive x -direction.

Lemma 7 *The following relationship holds*

$$\kappa = \frac{C_p \ \times C_{pp}}{|C_p|^3}. \tag{3}$$

Furthermore, κ is independent of the parametrization of γ.

Proof. We can write

$$\overrightarrow{T} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

Thus

$$\widetilde{C}_s = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

and

$$\begin{aligned} \widetilde{C}_{ss} &= \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \frac{d\theta}{ds} \\ &= \kappa \overrightarrow{N}. \end{aligned}$$

The formula (3) for κ follows by comparing the above equation with (2).

To show that κ is independent of the parametrization of γ suppose that $\widetilde{C}$ is a reparametrization of γ according to Definition 1. By Lemma 4,

$$\frac{C_p \ \times C_{pp}}{|C_p|^3} = \frac{\phi'^3 \widetilde{C}_q \times \widetilde{C}_{qq}}{\left|\phi' \widetilde{C}_q\right|^3} = \frac{\widetilde{C}_q \times \widetilde{C}_{qq}}{\left|\widetilde{C}_q\right|^3}.$$

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2 Invariant Signatures

We want to identify the differential signatures of a curve γ. These are intrinsic properties of the curve and are not affected by the position or orientation of the curve. Such quantities are useful for pattern recognition and object classification under partial occlusion.

2.1 Euclidean Invariants

By Euclidean invariants we mean quantities that are invariant under rotation and translation. If C is a parametrization of a curve γ, then $C + \mathbf{x}_0$ is a parametrization of its translation by a vector $\mathbf{x}_0 \in \mathbb{R}^2$ and $R(\theta_0)(C - \mathbf{z}_0) + \mathbf{z}_0$ is a parametrization of its rotation through an angle θ_0 about a point $\mathbf{z}_0 \in \mathbb{R}^2$.

Proposition 8 *The arclength and curvature are invariant under rotation and translation.*

Proof. The arclength of a rotated curve C is given by

$$\begin{aligned} s &= \int_a^p \left|(R(\theta_0)(C - \mathbf{z}_0) + \mathbf{z}_0)_p\right| d\sigma \\ &= \int_a^p |R(\theta_0) C_p| \, d\sigma \\ &= \int_a^p |C_p| \, d\sigma. \end{aligned}$$

Thus the arclength is invariant under rotation. Similarly we can show that it is invariant under translation. The invariance of κ under translation can be seen from

$$\kappa = \frac{C_p \times C_{pp}}{|C_p|^3}$$

since all quantities on the right involve differentiation with respect to the parameter p. Its invariance under rotation follows from the fact that the angle between C_p and C_{pp} as well as their lengths are preserved under rotation of both vectors through an angle θ_0. ■

Thus, the reconstruction of θ from κ (using $\kappa = \frac{d\theta}{ds}$) involves and arbitrary initial rotation θ_0. Similarly, the reconstruction of $\begin{pmatrix} x(s) \\ y(s) \end{pmatrix}$ from θ, through the equation

$$\frac{d}{ds}\begin{pmatrix} x(s) \\ y(s) \end{pmatrix} = \overrightarrow{T} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix},$$

also involves an arbitrary initial vector $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.

2.2 Affine Invariants of Planer Curves

An affine transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$\mathbf{y} = T\mathbf{x} = A\mathbf{x} + \mathbf{b},$$

where A is a constant 2×2 matrix such that $\det A > 0$ and $\mathbf{b}$ is a fixed vector in $\mathbb{R}^2$. If $\det A = 1$, the transformation preserves areas (show this). An affine transformation that preserves areas is called equi-affine. The rotations and translations of previous subsection are special cases of the equi-affine transformation.

Lemma 9 *Let γ be a curve parametrized by $C : [a,b] \to \mathbb{R}^2$. The form $C_p \times C_{pp}$ is invariant under equi-affine transformations.*

Proof. Suppose $T\mathbf{x} = A\mathbf{x} + \mathbf{b}$ is an equi-affine transformation. Define the parametrization $\widetilde{C}$ of $T\gamma$ by $\widetilde{C}(p) = TC(p) = AC(p) + \mathbf{b}$. We need to show that $\widetilde{C}_p \times \widetilde{C}_{pp} = C_p \times C_{pp}$. For this we have

$$\begin{aligned} \widetilde{C}_p \times \widetilde{C}_{pp} &= (AC + \mathbf{b})_p \times (AC + \mathbf{b})_{pp} \\ &= AC_p \times AC_{pp} \\ &= \det([AC_p \; AC_{pp}]) \\ &= \det(A[C_p \; C_{pp}]) \\ &= \det A \det[C_p \; C_{pp}] \\ &= C_p \times C_{pp}. \end{aligned}$$

■

The equi-affine arclength is a reparametrization $\widetilde{C}(v)$ of γ defined so that

$$\left| \widetilde{C}_v \times \widetilde{C}_{vv} \right| = 1. \qquad (4)$$

Lemma 10 *The reparametrization $\widetilde{C}$ is such that*

$$v = \phi(p) = \int_a^p |C_p(\sigma) \times C_{pp}(\sigma)|^{1/3} \, d\sigma. \tag{5}$$

Proof. Using Lemma 4, we have

$$1 = \left| \widetilde{C}_v \times \widetilde{C}_{vv} \right| = \frac{1}{\phi'^3} |C_p \times C_{pp}| .$$

Therefore,

$$\phi'^3 = |C_p \times C_{pp}| .$$

Hence,

$$\phi(p) = \int_a^p |C_p(\sigma) \times C_{pp}(\sigma)|^{1/3} \, d\sigma.$$

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Thus, the equi-affine arclength is given by

$$l_a(p) = \int_a^p |C_p(\sigma) \times C_{pp}(\sigma)|^{1/3} \, d\sigma.$$

One can easily show that the equation (4) is invariant under reparametrizations (verify). Thus, if γ is parametrized by arclength, then

$$v = \phi(s) = \int_0^s |C_s(\sigma) \times C_{ss}(\sigma)|^{1/3} \, d\sigma$$

Therefore,

$$\begin{aligned} \phi' &= |C_s \times C_{ss}|^{1/3} \\ &= \left| \overrightarrow{T} \times \kappa \overrightarrow{N} \right|^{1/3} \\ &= |\kappa|^{1/3} \left| \overrightarrow{T} \times \overrightarrow{N} \right|^{1/3} \\ &= |\kappa|^{1/3} . \end{aligned}$$

Thus, if C is a parametrization of γ by the equi-affine arclength and $\widetilde{C}$ is a parametrization by arclength, then $v = \phi(s)$ and$C(v) = \widetilde{C}(s) = \widetilde{C}(\phi(s))$. Therefore,

$$\begin{aligned} C_v &= \widetilde{C}_s \frac{ds}{dv} = \frac{1}{\phi'} \widetilde{C}_s \\ &= |\kappa|^{-1/3} \overrightarrow{T}, \end{aligned}$$

which, together with the chain rule, give

$$C_{vv} = \kappa^{1/3} \overrightarrow{N} - \frac{\kappa_s}{3 |\kappa|^{5/3}} \overrightarrow{T} .$$

Differentiating (the square of) (4) gives

$$C_v \times C_{vvv} = 0. \tag{6}$$

This means that C_v, C_{vvv} are collinear:

$$C_{vvv} = -\mu C_v.$$

From the above equation we get

$$\begin{aligned} C_{vv} \times C_{vvv} &= -\mu C_{vv} \times C_v \\ &= \pm\mu \end{aligned}$$

which means that is invariant under equi-affine transformations. μ is called the *affine curvature*. It is the simplest invariant of the curve γ.Also differentiating (6) gives

$$\begin{aligned} C_v \times C_{vvvv} + C_{vv} \times C_{vvv} &= 0 \\ C_v \times C_{vvvv} - \mu C_{vv} \times C_v &= 0 \\ C_v \times C_{vvvv} \pm \mu &= 0 \\ \mu &= \pm C_{vvvv} \times C_v. \end{aligned}$$

3 Calculus of Variation in Parametric Form

Suppose we are given a function $F : J \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ (i.e., $F = F(p, \mathbf{x}, \mathbf{y}), p \in J, \mathbf{x}, \mathbf{y} \in \mathbb{R}^2$). The problem of Calculus of variation (adapted to our settings) is the one of finding a curve γ that immunizes the expression

$$J(\gamma) = \int_{p_1}^{p_2} F(p, C(p), C_p(p))\, dp, \tag{7}$$

where C is a parametrization of γ. A minimizing curve with a parametrization $C(p) = \begin{pmatrix} x(p) \\ y(p) \end{pmatrix}$ satisfies the Euler-Lagrange equation

$$\frac{d}{dp} F_{\mathbf{y}}(p, C(p), C_p(p)) - F_{\mathbf{x}}(p, C(p), C_p(p)) = 0.$$

The following example should clarify this notation

Example Let

$$F(p, X, \mathbf{y}) = p^2 \left(|\mathbf{x}|^2 + |\mathbf{y}|^2\right)$$

with $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$. Then $F_{\mathbf{x}}(p, \mathbf{x}, \mathbf{y}) = \nabla_{\mathbf{x}} F(p, \mathbf{x}, \mathbf{y}) = 2p^2\mathbf{x}$ and $F_{\mathbf{y}}(p, \mathbf{x}, \mathbf{y}) = \nabla_{\mathbf{y}} F(p, \mathbf{x}, \mathbf{y}) = 2p^2\mathbf{y}$. Setting $\mathbf{x} = C, \mathbf{y} = C_p$ and substituting in the Euler-Lagrange equation, we obtain

$$\frac{d}{dp}\left(2p^2 C_p\right) - 2p^2 C = 0$$

which reduces to the second order system

$$pC_{pp} + 2C_p = pC.$$

Observe that this is a system of two equations that can be solved separately for each component.

Exercise Repeat the above example for $F(p, \mathbf{x}, \mathbf{y}) = e^{-p\langle \mathbf{x}, \mathbf{y}\rangle}$.

Naturally, we require that the expression $J(\gamma)$ to be independent of the parametrization of γ. Let's examine what class of functions F will satisfy this requirement. For this purpose, assume C and $\widetilde{C}$ are parametrizations of γ according to Definition 1. Then the required independence of parametrization means

$$\begin{aligned}\int_{p_1}^{p_2} F(p, C(p), C_p(p))\, dp &= \int_{q_1}^{q_2} F\left(q, \widetilde{C}(q), \widetilde{C}_q(q)\right) dq \\ &= \int_{p_1}^{p_2} F\left(\phi(p), C(p), \frac{1}{\phi'(p)} C_p(p)\right) \phi'(p)\, dp.\end{aligned}$$

Therefore, we must have

$$F(p, C(p), C_p(p)) = F\left(\phi(p), C(p), \frac{1}{\phi'(p)} C_p(p)\right) \phi'(p).$$

In the case $q = \phi(p) = p + c$, we get

$$F(p + c, C(p), C_p(p)) = F(p, C(p), C_p(p)).$$

which means that F must be independent of p, and in the case and $q = \phi(p) = cp$, we get

$$F\left(C(p), \frac{1}{c} C_p(p)\right) c = F(C(p), C_p(p)),$$

which implies that

$$F(C(p), cC_p(p)) = cF(C(p), C_p(p)).$$

for all positive numbers c. That is to say, F must also be positive homogenous with respect to its second argument. For convenience, we will write the above equation and the Euler-Lagrange equation as

$$\begin{aligned} F(\mathbf{x}, c\mathbf{x}') &= cF(\mathbf{x}, \mathbf{x}') \qquad (8) \\ \frac{d}{dp} F_{\mathbf{y}}(\mathbf{x}, \mathbf{x}') &= F_{\mathbf{x}}(\mathbf{x}, \mathbf{x}'). \qquad (9)\end{aligned}$$

Lemma 11 *For the functional (7) to be independent of the parametrization, F must be independent of p and we must have*

$$F_{\mathbf{yy}}(\mathbf{x}, \mathbf{x}')\, \mathbf{x}' = 0. \qquad (10)$$

Here $F_{\mathbf{yy}}$ is the matrix

$$F_{\mathbf{yy}} = \begin{bmatrix} F_{y_1 y_1} & F_{y_1 y_2} \\ F_{y_2 y_1} & F_{y_2 y_2} \end{bmatrix}$$

Proof. We have already seen that F must be independent of p. To show (10) differentiate equation (8) with respect to c and set $c = 1$ to get

$$F_{\mathbf{y}}\mathbf{x}' = F. \tag{11}$$

Differentiate once again with respect to $\mathbf{x}'$ to get

$$F_{\mathbf{yy}}\mathbf{x}' + F_{\mathbf{y}} = F_{\mathbf{y}}$$

or

$$F_{\mathbf{yy}}\mathbf{x}' = 0.$$

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Theorem 12 *The curve γ that minimizes the functional (7) satisfies the equation*

$$F_{\mathbf{yy}}\left(\mathbf{x}, \mathbf{x}'\right)\mathbf{x}'' = 0. \tag{12}$$

and κ is the curvature.

Proof. The Euler-Lagrange equation (9) may be rewritten as

$$F_{\mathbf{yx}}\mathbf{x}' + F_{\mathbf{yy}}\mathbf{x}'' = F_{\mathbf{x}}.$$

To eliminate $F_{\mathbf{x}}$, we differentiate both sides of equation (11) with respect to $\mathbf{x}$. This yields

$$F_{\mathbf{yx}}\mathbf{x}' = F_{\mathbf{x}}.$$

Substituting in the previous equation and simplifying, we get

$$F_{\mathbf{yy}}\mathbf{x}'' = 0.$$

■

Exapmle Let's verify that the shortest curve between two points Q_1, Q_2 in $\mathbb{R}^2$ is the straight line. For this we need to minimize the expression for the Euclidean arclength of a curve. That is

$$J(\gamma) = \int_a^b |C_p|\, d\sigma.$$

Therefore, $F(\mathbf{x}, \mathbf{x}') = |\mathbf{x}'|$, or $F(\mathbf{x}, \mathbf{y}) = F(\mathbf{y}) = |\mathbf{y}|$, for which

$$F_{\mathbf{yy}}(\mathbf{x}') = \frac{1}{|\mathbf{x}'|^3}\begin{pmatrix} x_2'^2 & -x_1'x_2' \\ -x_1'x_2' & x_1'^2 \end{pmatrix},$$

where we wrote

$$\mathbf{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}.$$

Observe that

$$F_{\mathbf{yy}}(\mathbf{x}')\mathbf{x}' = \frac{1}{|\mathbf{x}'|^3}\begin{pmatrix} x_2'^2 & -x_1'x_2' \\ -x_1'x_2' & x_1'^2 \end{pmatrix}\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = 0.$$

The Euler-Lagrange equation (12) gives

$$\frac{1}{|\mathbf{x}'|^3}\begin{pmatrix} x_2'^2 & -x_1'x_2' \\ -x_1'x_2' & x_1'^2 \end{pmatrix}\begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} = \frac{1}{|\mathbf{x}'|^3}\begin{pmatrix} -x_1'x_2'x_2'' + x_1''x_2'^2 \\ -x_1'x_2'x_1'' + x_2''x_1'^2 \end{pmatrix} = 0.$$

The first equation in the above system can be simplified as follows

$$\begin{aligned} \frac{1}{|\mathbf{x}'|^3}x_2'\left(x_1'x_2'' - x_1''x_2'\right) &= 0 \\ x_2'\kappa &= 0. \end{aligned}$$

Similarly,

$$x_1'\kappa = 0.$$

Since x_1' and x_2' cannot be zero simultaneously, we must have

$$\kappa = 0.$$

Then

$$C_{ss} = \kappa\overrightarrow{N} = 0.$$

Therefore,

$$C(s) = P_0 s + P_1,\ \ P_0, P_1 \in \mathbb{R}^2,$$

which is a parametrization of a straight line.

Exercise Repeat the above example for the equi-affine arclength

$$J(\gamma) = \int_a^b |C_p \times C_{pp}|^{1/3}\, d\sigma.$$

4 Surfaces in $\mathbb{R}^3$

A surface is intuitively thought of as composed of a finite number of pieces, called surface patches, each of which being locally like $\mathbb{R}^2$. Examples of surfaces are sheets, surfaces of spheres, doughnuts,... etc. A surface is also termed a 2-manifold. We begin by defining regular transformations.

Definition 13 *Suppose U is an open subset of $\mathbb{R}^2$. A regular transformation $\sigma : U \to \mathbb{R}^3$ has the following properties.*

1. *σ is of class C^1 (i.e., σ' is continuous).*
2. *σ is univalent (i.e., 1:1).*
3. *For any $\mathbf{u} \in U$, $\sigma'(\mathbf{u})$ has rank 2.*

Definition 14 *A subset M of $\mathbb{R}^3$ is called a regular surface patch in $\mathbb{R}^3$ if there is an open subset U of $\mathbb{R}^2$ and a regular transformation $\sigma : U \to \mathbb{R}^3$ such that $\sigma(U) = M$. We will call σ a parametrization of M .*

Example Let $M = \{(x, y, z) \in \mathbb{R}^3 : z^2 = x^2 + y^2, z > 0\}$ (M is the upper half of a cone with axis along the z-axis), $U = \{(u, v) \in \mathbb{R}^2 : u > 0, 0 < v < 2\pi\}$ and $\sigma : U \to M$ be the transformation

$$\sigma(u, v) = \begin{pmatrix} u \cos v \\ u \sin v \\ u \end{pmatrix}.$$

Then

$$\sigma'(u, v) = \begin{pmatrix} \cos v & -u \sin v \\ \sin v & u \cos v \\ 1 & 0 \end{pmatrix}.$$

It is straightforward to verify that σ is a regular transformation and that $\sigma(U) = M$.

A regular curve in $\mathbb{R}^3$ is parametrized by a mapping $\gamma : J = [a, b] \to \mathbb{R}^3$ such that $\gamma \in C^1(J)$, γ is univalent and $\gamma'(t) \neq 0$ for all $t \in J$. Suppose M is a regular surface patch in $\mathbb{R}^3$ parametrized by $\sigma : U \to M$. Any curve on M has a parametrization of the form

$$\begin{aligned} \gamma(t) &= \sigma(\mathbf{u}(t)) \\ &= \sigma(u(t), v(t)), \ t \in J \end{aligned}$$

for some interval J, where $\mathbf{u}(t) = (u(t), v(t)) : J \to U$ is a regular curve in U. Observe that $\gamma'(t) = \sigma_u u' + \sigma_v v' = \sigma' \mathbf{u}'$ and since σ' has rank 2 and $\mathbf{u}' \neq 0$, $\gamma'(t) \neq 0$ for all $t \in J$. Also γ is univalent and C^1.

Definition 15 *Let M be a regular surface patch in $\mathbb{R}^3$ parametrized by $\sigma : U \to M$ and let $\mathbf{x}_0 \in M$ be such that $\sigma(\mathbf{u}_0) = \mathbf{x}_0$. A vector $\mathbf{h} \in \mathbb{R}^3$ is called a tangent vector to M at $\mathbf{x}_0$ if there is a regular curve $\gamma(t) = \sigma(u(t), v(t)) : J \to M$ passing through $\mathbf{x}_0$ (i.e., $\gamma(t_0) = \sigma(\mathbf{u}_0) = \mathbf{x}_0$) such that $\gamma'(t_0) = \mathbf{h}$. The set of all tangent vectors to M at $\mathbf{x}_0$ is called the tangent plane at $\mathbf{x}_0$ and is denoted by $T_M(\mathbf{x}_0)$.*

Proposition 16 *Let M be a regular surface patch in $\mathbb{R}^3$ parametrized by $\sigma : U \to M$ with $\sigma : \mathbf{u}_0 \in U \to \mathbf{x}_0 \in M$. Then $\sigma'(\mathbf{u}_0)$ spans $T_M(\mathbf{x}_0)$.*

Proof. Let $\mathbf{h} \in T_M(\mathbf{x}_0)$, by definintion, there is a regular curve $\gamma(t) = \sigma(u(t), v(t)) : J \to M$, with $\gamma(t_0) = \sigma(\mathbf{u}_0) = \mathbf{x}_0$ such that $\gamma'(t_0) = \mathbf{h}$. Therefore,

$$\begin{aligned} \mathbf{h} &= \gamma'(t_0) = \sigma_u(\mathbf{u}_0) u'(t_0) + \sigma_v(\mathbf{u}_0) v'(t_0) \\ &= \sigma'(\mathbf{u}_0) \mathbf{u}'(t_0). \end{aligned}$$

Therefore, $\mathbf{h}$ is a linear combination of the columns of $\sigma'(\mathbf{u}_0)$. On the other hand, let $\mathbf{h}_1 = \sigma'(\mathbf{u}_0) \mathbf{e}_1 = \sigma_u(\mathbf{u}_0)$ and $\mathbf{h}_2 = \sigma'(\mathbf{u}_0) \mathbf{e}_2 = \sigma_v(\mathbf{u}_0)$. Since $\sigma'(\mathbf{u}_0)$ has rank 2, $\mathbf{h}_1, \mathbf{h}_2$ are linearly independent. We will show now that $\mathbf{h}_1, \mathbf{h}_2 \in T_M(\mathbf{x}_0)$ .Since U is open and $\mathbf{u}_0 \in U$, there is an $r > 0$ such that $B(\mathbf{u}_0, r) \subset U$. It follows that the line "curve" $(\mathbf{u}_0 - r\mathbf{e}_1, \mathbf{u}_0 + r\mathbf{e}_1) \subset B(\mathbf{u}_0, r) \subset U$. Define the function $\mathbf{u} : (-r, r) \to U$ by

$$\mathbf{u}(t) = \mathbf{u}_0 + t\mathbf{e}_1$$

and the curve γ on M by

$$\begin{aligned}\gamma(t) &= \sigma(\mathbf{u}(t)) \\ &= \sigma(\mathbf{u}_0 + t\mathbf{e}_1).\end{aligned}$$

Then

$$\begin{aligned}\gamma(0) &= \sigma(\mathbf{u}_0) = \mathbf{x}_0, \\ \gamma'(t) &= \sigma'(\mathbf{u}_0 + t\mathbf{e}_1)\mathbf{e}_1, \\ \gamma'(0) &= \sigma'(\mathbf{u}_0)\mathbf{e}_1 = \mathbf{h}_1.\end{aligned}$$

Therefore, $\mathbf{h}_1 \in T_M(\mathbf{x}_0)$. Similarly we can show that $\mathbf{h}_2 \in T_M(\mathbf{x}_0)$. ■

Corollary 17 *$T_M(\mathbf{x}_0)$ is the image of $\mathbb{R}^2$ under the linear transformation* $\mathbf{k} \in \mathbb{R}^2 :\to \sigma'(\mathbf{u}_0)\mathbf{k} \in \mathbb{R}^3$.

Proof. Let $\mathbf{k} \in \mathbb{R}^2$ and write $\mathbf{k} = \alpha\mathbf{e}_1 + \beta\mathbf{e}_2$. Then $\sigma'(\mathbf{u}_0)\mathbf{k} = \alpha\sigma'(\mathbf{u}_0)\mathbf{e}_1 + \beta\sigma'(\mathbf{u}_0)\mathbf{e}_2 = \alpha\mathbf{h}_1 + \beta\mathbf{h}_2 \in T_M(\mathbf{x}_0)$. On the other hand, for any $\mathbf{h} \in T_M(\mathbf{x}_0)$ we can write $\mathbf{h} = \alpha\mathbf{h}_1 + \beta\mathbf{h}_2$ since $\sigma'(\mathbf{u}_0)$ spans $T_M(\mathbf{x}_0)$. Therefore,

$$\begin{aligned}\mathbf{h} &= \alpha\sigma'(\mathbf{u}_0)\mathbf{e}_1 + \beta\sigma'(\mathbf{u}_0)\mathbf{e}_2 \\ &= \sigma'(\mathbf{u}_0)(\alpha\mathbf{e}_1 + \beta\mathbf{e}_2) \\ &= \sigma'(\mathbf{u}_0)\mathbf{k},\end{aligned}$$

where $\mathbf{k} = (\alpha\mathbf{e}_1 + \beta\mathbf{e}_2)$. ■

We turn next to considering integrals on surfaces and the computation of surface area. Let M be a regular surface patch in $\mathbb{R}^3$ parametrized by $\sigma : U \to M$. Let $A = \sigma(B)$, where $B \subset U$ is measurable (i.e., has a finite area). Then the area of A is defined by

$$V_2(A) = \int_B \mathcal{I}(\sigma(\mathbf{u}))\, dV_2(\mathbf{u}),$$

where

$$\mathcal{I}(\sigma(\mathbf{u})) = |\sigma_u(\mathbf{u}) \times \sigma_v(\mathbf{u})|$$

and $dV_2(\mathbf{u})$ is either $dudv$ or $dvdu$ depending on the order of integration.

Example For the previous example,

$$\begin{aligned}\mathcal{I}(\sigma(u,v)) &= \left\| \begin{bmatrix} \cos v \\ \sin v \\ 1 \end{bmatrix} \times \begin{bmatrix} -u\sin v \\ u\cos v \\ 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -u\cos v \\ -u\sin v \\ u \end{bmatrix} \right\| \\ &= \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2} = \sqrt{2}u.\end{aligned}$$

Take $B = \{(u,v) \in \mathbb{R}^2 : a < u < b, 0 < v < \pi\}$. Then $A = \sigma(B)$ is the portion of the cone $z^2 = x^2 + y^2$ between the planes $z = a$ and $z = b$ for which $y > 0$. We have

$$\begin{aligned}V_2(A) &= \int_0^\pi \int_a^b \sqrt{2}u \; dudv \\ &= \frac{\pi}{\sqrt{2}}\left(b^2 - a^2\right).\end{aligned}$$

In analogy with curves in $\mathbb{R}^2$, we discuss the idea of a reparametrization of a surface patch M. Let $\phi : U \to \mathbb{R}^2$ be a regular transformation of class C^1 (i.e., ϕ satisfies all the conditions of Definition 13 with σ replaced by ϕ and $\mathbb{R}^3$ replaced by $\mathbb{R}^2$). We will call ϕ a regular flat transformation.

Definition 18 *Let $\phi : U \to \mathbb{R}^2$ be a regular flat transformation and set $\widetilde{U} = \phi(U)$. Suppose M is a regular surface patch in $\mathbb{R}^3$ parametrized by σ. The (regular) transformation $\widetilde{\sigma} : \widetilde{U} \to M$ defined by*

$$\sigma(\mathbf{u}) = \widetilde{\sigma} \circ \phi(\mathbf{u}) = \widetilde{\sigma}(\mathbf{v}),$$

where $\mathbf{u} = (u, v)$ stands for a variable in U and $\mathbf{v} = (r, s)$ stands for a variable in $\widetilde{U}$, that is $\mathbf{v} = \phi(\mathbf{u})$, will be called a reparametrization of M.

Before proving a "reparametrization" formula for surfaces we need the following lemma.

Lemma 19 *Let $A : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation and write $A = [\mathbf{a}_1\ \mathbf{a}_2]$, where $\mathbf{a}_1$, $\mathbf{a}_2$ are the columns of A. For any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$,*

$$A\mathbf{x} \times A\mathbf{y} = (\mathbf{a}_1 \times \mathbf{a}_2)(\mathbf{x} \times \mathbf{y}).$$

Proof. Write $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Then $A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2$ and $A\mathbf{y} = y_1\mathbf{a}_1 + y_2\mathbf{a}_2$.

$$\begin{aligned} A\mathbf{x} \times A\mathbf{y} &= (x_1\mathbf{a}_1 + x_2\mathbf{a}_2) \times (y_1\mathbf{a}_1 + y_2\mathbf{a}_2) \\ &= x_1 y_2 (\mathbf{a}_1 \times \mathbf{a}_2) + x_2 y_1 (\mathbf{a}_2 \times \mathbf{a}_1) \\ &= (x_1 y_2 - x_2 y_1)(\mathbf{a}_1 \times \mathbf{a}_2) \\ &= (\mathbf{a}_1 \times \mathbf{a}_2)(\mathbf{x} \times \mathbf{y}). \end{aligned}$$

■

Proposition 20 *Let M be a regular surface patch in $\mathbb{R}^3$. Let $\sigma, \widetilde{\sigma}$ be parametrizations of M according to definition 18. Then*

$$\mathcal{I}(\sigma(\mathbf{u})) = \mathcal{I}(\widetilde{\sigma}(\mathbf{v}))\,|J\phi(\mathbf{u})|,$$

where

$$J\phi(\mathbf{u}) = \det \phi'(\mathbf{u}).$$

Proof. Since $\sigma(\mathbf{u}) = \widetilde{\sigma} \circ \phi(\mathbf{u}) = \widetilde{\sigma}(\mathbf{v})$,

$$\begin{aligned} \sigma_{\mathbf{u}}(\mathbf{u}) &= \widetilde{\sigma}_{\mathbf{v}} \circ \phi(\mathbf{u})\, \phi'(\mathbf{u}) \\ &= \widetilde{\sigma}_{\mathbf{v}}(\mathbf{v})\, \phi'(\mathbf{u}). \end{aligned}$$

In otherwords, we have

$$\begin{aligned} \sigma_u(\mathbf{u}) &= \widetilde{\sigma}_{\mathbf{v}}(\mathbf{v})\, \phi_u(\mathbf{u}), \\ \sigma_v(\mathbf{u}) &= \widetilde{\sigma}_{\mathbf{v}}(\mathbf{v})\, \phi_v(\mathbf{u}). \end{aligned}$$

Observe also that $\widetilde{\sigma}_{\mathbf{v}} = [\widetilde{\sigma}_r \ \widetilde{\sigma}_s]$. Now, using Lemma 19,

$$\begin{aligned}
\mathcal{I}(\sigma(\mathbf{u})) &= |\sigma_u(\mathbf{u}) \times \sigma_v(\mathbf{u})| \\
&= |\ \widetilde{\sigma}_{\mathbf{v}}\phi_u \times \widetilde{\sigma}_{\mathbf{v}}\phi_v| \\
&= |\widetilde{\sigma}_r \times \widetilde{\sigma}_s|\ |\phi_u \times \phi_v| \\
&= \mathcal{I}(\widetilde{\sigma}(\phi(\mathbf{u})))\,|J\phi(\mathbf{u})| \\
&= \mathcal{I}(\widetilde{\sigma}(\mathbf{v}))\,|J\phi(\mathbf{u})|.
\end{aligned}$$

■

We next define what we mean by integrating a function f over a regular surface patch in $\mathbb{R}^3$.

Definition 21 *Let M be a regular surface patch in $\mathbb{R}^3$, $A \subset M$ be of finite area and $f : A \to \mathbb{R}$ be continuous. The integral of f over A is defined by*

$$\int_A f(\mathbf{x})\,dV_2(\mathbf{x}) = \int_B f(\sigma(\mathbf{u}))\,\mathcal{I}(\sigma(\mathbf{u}))\,dV_2(\mathbf{u}),$$

where $A = \sigma(B)$, provided that $(f \circ \sigma)\,\mathcal{I}(\sigma)$ is integrable over B.

We also recall the following change of variable formula from advanced calculus. Suppose $U, \widetilde{U}$ are subsets of $\mathbb{R}^2$ and $\phi : U \to \widetilde{U}$ is a flat reguar transformation. Let $B \subset U, \widetilde{B} \subset \widetilde{U}$ be such that $\widetilde{B} = \phi(B)$. If $g : B \to \mathbb{R}$ is an integrable function, then

$$\int_{\widetilde{B}} g(\mathbf{v})\,dV_2(\mathbf{v}) = \int_B g(\phi(\mathbf{u}))\,|J\phi(\mathbf{u})|\,dV_2(\mathbf{u}).$$

The following corollary states that the integral of a function f over a surface is independent of the parametrization of the surface.

Corollary 22 *Suppose M is a regular surface patch in $\mathbb{R}^3$ parametrized by σ and $\widetilde{\sigma}$ according to definition 18. Suppose further that $B \subset U$ has finite measure and $\widetilde{B} = \phi(B)$. Then*

$$\begin{aligned}
\int_A f(\mathbf{x})\,dV_2(\mathbf{x}) &= \int_{\widetilde{B}} f(\widetilde{\sigma}(\mathbf{v}))\,\mathcal{I}(\widetilde{\sigma}(\mathbf{v}))\,dV_2(\mathbf{v}) \\
&= \int_B f(\widetilde{\sigma}(\phi(\mathbf{u})))\,\mathcal{I}(\widetilde{\sigma}(\phi(\mathbf{u})))\,|J\phi(\mathbf{u})|\,dV_2(\mathbf{u}) \\
&= \int_B f(\sigma(\mathbf{u}))\,\mathcal{I}(\sigma(\mathbf{u}))\,dV_2(\mathbf{u}),
\end{aligned}$$

where $A = \sigma(B) = \widetilde{\sigma}\left(\widetilde{B}\right)$.

4.1 The first fundamental form and Geodesic distance

Let M be a regular surface patch in $\mathbb{R}^3$ parametrized by $\sigma : U \to M$. Suppose $\gamma(t) = \sigma(u(t), v(t))$, $t \in J = [a, b]$ is a regular curve on M. The arclength of γ is defined by

$$\begin{aligned} l(\gamma) &= \int_a^b |\gamma'(t)|\, dt \\ &= \int_a^b |\sigma_u u' + \sigma_v v'|\, dt. \end{aligned}$$

Example For the example of the conic surface, let $\mathbf{u} : J = \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \to U$ be given by $\mathbf{u}(t) = (u(t), v(t)) = (t, t)$ and let

$$\begin{aligned} \gamma(t) &= \sigma(u(t), v(t)) \\ &= \sigma(t, t) \\ &= \begin{bmatrix} t\cos t \\ t\sin t \\ t \end{bmatrix}. \end{aligned}$$

Then

$$l(\gamma) = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{2}t dt = \frac{3\sqrt{2}}{32}\pi^2.$$

Observe that the endpoints of γ are $\gamma\left(\frac{\pi}{4}\right) = \begin{pmatrix} \sqrt{2}\pi/8 \\ \sqrt{2}\pi/8 \\ \pi/4 \end{pmatrix}$ and $\gamma\left(\frac{\pi}{2}\right) = \begin{pmatrix} 0 \\ \pi/2 \\ \pi/2 \end{pmatrix}$.

Let's formally write

$$\begin{aligned} ds^2 &= |\sigma_u u' + \sigma_v v'|^2 dt^2 \\ &= \langle \sigma_u u' + \sigma_v v', \sigma_u u' + \sigma_v v' \rangle dt^2 \\ &= \left(|\sigma_u|^2 u'^2 + 2\langle \sigma_u, \sigma_v \rangle u'v' + |\sigma_v|^2 v'^2 \right) dt^2 \\ &= |\sigma_u|^2 du^2 + 2\langle \sigma_u, \sigma_v \rangle dudv + |\sigma_v|^2 dv^2 \\ &= Edu^2 + 2Fdudv + Gdv^2, \end{aligned}$$

where $E = |\sigma_u|^2$, $F = \langle \sigma_u, \sigma_v \rangle$ and $G = |\sigma_v|^2$. The above equation is a formal definition, which is used (as far as we are concerned) to keep track of the quantities E, F, G.

Definition 23 *The expression*

$$Edu^2 + 2Fdudv + Gdv^2$$

is called the first fundamental form of M.

Suppose $\mathbf{h}_1, \mathbf{h}_2 \in T_M(\mathbf{x}_0)$ then $\mathbf{h}_1, \mathbf{h}_2$ are linear combinations of $\sigma_u(\mathbf{x}_0), \sigma_v(\mathbf{x}_0)$. Hence we can write

$$\begin{aligned}\mathbf{h}_1 &= \xi_1\sigma_u + \eta_1\sigma_v, \\ \mathbf{h}_2 &= \xi_2\sigma_u + \eta_2\sigma_v.\end{aligned}$$

It is easy to check that

$$\begin{aligned}\langle \mathbf{h}_1, \mathbf{h}_2 \rangle &= E\xi_1\xi_2 + F(\xi_1\eta_2 + \eta_1\xi_2) + G\eta_1\eta_2 \\ &= T_1^t \mathcal{F}_I T_{2,}\end{aligned}$$

where

$$\begin{aligned}\mathcal{F}_I &= \begin{bmatrix} E & F \\ F & G \end{bmatrix}, \\ T_1 &= \begin{bmatrix} \xi_1 \\ \eta_1 \end{bmatrix}, T_2 = \begin{bmatrix} \xi_2 \\ \eta_2 \end{bmatrix}.\end{aligned}$$

In particular, if $\mathbf{h}_1 = \mathbf{h}_2 = \mathbf{h} =$ a unit vector, then

$$T^t \mathcal{F}_I T = 1$$

and if $\mathbf{h}_1, \mathbf{h}_2$ are orthogonal, then

$$T_1^t \mathcal{F}_I T_2 = 0.$$

Now suppose that $\mathbf{h}_1, \mathbf{h}_2 \in T_M(\mathbf{x}_0)$ are two orthonormal vectors and let

$$A = [T_1 \ T_2],$$

then

$$A^t \mathcal{F}_I A = \begin{bmatrix} T_1^t \mathcal{F}_I T_1 & T_1^t \mathcal{F}_I T_2 \\ T_2^t \mathcal{F}_I T_1 & T_2^t \mathcal{F}_I T_2 \end{bmatrix} = I_2.$$

Definition 24 *For $\mathbf{x}, \mathbf{y} \in M$, the geodesic distance $d(\mathbf{x}, \mathbf{y})$ is defined by*

$$d(\mathbf{x}, \mathbf{y}) = \inf\{l(\gamma) : \gamma \textit{ is a curve on } M \textit{ with endpoints } \mathbf{x} \textit{ and } \mathbf{y}\}.$$

4.2 The Second Fundamental Form and Surface Curvatures

Suppose γ is a curve on a regular surface patch M in $\mathbb{R}^3$ parametrized by σ. Then

$$\gamma' = \sigma_u u' + \sigma_v v'$$

and

$$\gamma'' = \sigma_{uu} u'^2 + 2\sigma_{uv} u'v' + \sigma_{vv} v'^2 + \text{a tangent component.}$$

In particular, if $\overrightarrow{N}$ is the unit vector normal to the surface M at $\mathbf{x}$ then

$$\left\langle \gamma'', \overrightarrow{N} \right\rangle = Lu'^2 + 2Mu'v' + Nv'^2, \tag{13}$$

where

$$L = \left\langle \sigma_{uu}, \overrightarrow{N} \right\rangle, \ M = \left\langle \sigma_{uv}, \overrightarrow{N} \right\rangle, \ N = \left\langle \sigma_{vv}, \overrightarrow{N} \right\rangle$$

Definition 25 *The expression*

$$Ldu^2 + 2Mdudv + Ndv^2$$

is called the second fundamental form of M.

In analogy with the first fundamental form, we associate the matrix

$$\mathcal{F}_{II} = \begin{bmatrix} L & M \\ M & N \end{bmatrix}$$

with the second fundamental form.

Suppose that γ is a curve on M parametrized by arclength so that γ' is a unit vector. As we saw before, γ'' is orthogonal to γ'. Therefore, γ'' is parallel to the plane Π orthogonal to γ'. An orthonormal basis for Π can be taken as the unit normal $\overrightarrow{N}$ and $\overrightarrow{N} \times \gamma'$. Hence, γ'' is a linear combination of $\overrightarrow{N}$ and $\overrightarrow{N} \times \gamma'$

$$\gamma'' = \kappa_n \overrightarrow{N} + \kappa_g \overrightarrow{N} \times \gamma'.$$

κ_n and κ_g are called the *normal curvature* and the *geodesic curvature* of γ, respectively. It follows that

$$\kappa^2 = |\gamma''|^2 = \kappa_n^2 + \kappa_g^2, \tag{14}$$

where κ is the unsigned curvature of γ.

Denoting by $\mathbf{n}$ the unit vector in the directionof γ'', we have

$$\gamma'' = \kappa \mathbf{n}.$$

Therefore,

$$\kappa \mathbf{n} = \kappa_n \overrightarrow{N} + \kappa_g \overrightarrow{N} \times \gamma'$$

and if we take the inner product on both sides with $\overrightarrow{N}$ we get

$$\kappa_n = \kappa \left\langle \mathbf{n}, \overrightarrow{N} \right\rangle = \kappa \cos \psi, \tag{15}$$

where ψ is the angle between $\mathbf{n}, \overrightarrow{N}$. From (14) we get that

$$\kappa_g = \pm \kappa \sin \psi.$$

This means that κ_n is always well defined and κ_g is well defined up to a sign. Returning back to equation (15), we get, using also equation (13)

$$\begin{aligned} \kappa_n &= \kappa \left\langle \mathbf{n}, \overrightarrow{N} \right\rangle = \left\langle \kappa \mathbf{n}, \overrightarrow{N} \right\rangle \\ &= \left\langle \boldsymbol{\gamma}'', \overrightarrow{N} \right\rangle = Lu'^2 + 2Mu'v' + Nv'^2. \end{aligned}$$

Definition 26 *The principal curvatures of a surface patch are the roots of the equation*

$$\det\left(\mathcal{F}_{II} - \lambda \mathcal{F}_I\right),$$

i.e. the solutions of

$$\begin{vmatrix} L - \lambda E & M - \lambda F \\ M - \lambda F & N - \lambda G \end{vmatrix} = 0.$$

The principal curvatures κ_1 and κ_2 are also known as the *generalised eigenvalues* of the matrix $\mathcal{F}_{II}$ relative to the matrix $\mathcal{F}_I$. We will see in Proposition 27 that κ_1 and κ_2 are always real. Suppose that $T = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$ is a generalised eigenvector, that is, a solution of

$$\left(\mathcal{F}_{II} - \lambda \mathcal{F}_I\right) T = 0.$$

The corresponding tangent vector, $\mathbf{h} = \sigma' T = \xi \sigma_u + \eta \sigma_v$ to the surface M is called a *principal vector* corresponding to the principal curvature λ.

Proposition 27 *Let κ_1 and κ_2 be the principal curvatures at a point $\mathbf{x}$ of a survace patch M parametrized by σ. Then*

1. *κ_1 and κ_2 are real.*
2. *If $\kappa_1 = \kappa_2 = \lambda$, say, then $\mathcal{F}_{II} = \lambda \mathcal{F}_I$ and (hence) every tangent vector to M at $\mathbf{x}$ is a principal vector.*
3. *If $\kappa_1 \neq \kappa_2$ then any two (nonzero) principal vectors $\mathbf{h}_1$ and $\mathbf{h}_2$ corresponding to κ_1 and κ_2, respectively, are orthogonal.*

Proof. We prove only the first part of this proposition, leaving the other two parts as exercises. Let $\mathbf{h}_1 = \sigma' T_1$ and $\mathbf{h}_2 = \sigma' T_2$ be any two orthonormal tangent vectors to M at $\mathbf{x}$. Since σ' has rank 2, T_1 and T_2 are linearly independent. Consequently, the matrix $A = [T_1 \; T_2]$ has a nonzero determinant. We saw previously that $A^t \mathcal{F}_I A = I_2$. Therefore,

$$\begin{aligned} \det\left(\mathcal{F}_{II} - \lambda \mathcal{F}_I\right) &= \frac{1}{(\det A)^2} \det A^t \det\left(\mathcal{F}_{II} - \lambda \mathcal{F}_I\right) \det A \\ &= \frac{1}{(\det A)^2} \det\left(A^t \mathcal{F}_{II} A - \lambda I_2\right). \end{aligned}$$

Thus, the solutions of $\det\left(\mathcal{F}_{II} - \lambda \mathcal{F}_I\right) = 0$ are the eigenvalues of the matrix $A^t \mathcal{F}_{II} A$. These eigenvalues are real since $A^t \mathcal{F}_{II} A$ is symmetric. ■

Corollary 28 *(Euler's Theorem) Let γ be a curve on a surface patch M and let κ_1 and κ_2 be the principal curvatures with corresponding principal (nonzero) vectors $\mathbf{h}_1$ and $\mathbf{h}_2$ then the normal curvature of γ is*

$$\kappa_n = \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta$$

where θ is the angle between γ' and $\mathbf{h}_1$.

Proof. (Exercise). ■

Corollary 29 *The principal curvatures at a point of a surface are the maximum and minimum valures of the normal curvature of all curves on the surface that pass through the point. Moreover, the principal vectors are the tangent vectors of the curves that give these maximum and minimum values.*

Proof. (Exercise). ■

Curves γ with the geodesic curvature $\kappa_g = 0$ are called geodesics. An important case is when γ is the intersection of the surface M with a plane Π that is orthogonal to the tangent plane of M at every point of γ. We can show that $\kappa_g = 0$ in this case.

Definition 30 *The Gaussian curvature is defined by*

$$K = \kappa_1 \kappa_2$$

and the mean curvature is defined by

$$H = \frac{1}{2} \left(\kappa_1 + \kappa_2 \right).$$

Observe that κ_1 and κ_2 are the roots of the quadratic equation

$$\kappa^2 - 2H\kappa + K = 0.$$