The Osher-Sethian Level Set Method

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1 Curves as Level Sets

In this chapter we introduce the level set method for curve and surface evolution. The origin of this method is the simple observation made by Osher and Sethian that a curve γ in $\mathbb{R}^2$ (or a surface σ in $\mathbb{R}^3$) may be regarded as a level curve (surface) of the graph of a function defined in higher dimension. Recall that the graph of a function $\phi(x, y)$ of two variables is the set

$$\left\{(x, y, z) \in \mathbb{R}^3 : z = \phi(x, y), (x, y) \in D(\phi)\right\}$$

and the graph of a function $\phi(x, y, z)$ of three variables is the set

$$\left\{(x, y, z, w) \in \mathbb{R}^4 : w = \phi(x, y, z), (x, y, z) \in D(\phi)\right\}$$

and so on. The c-level curve (surface) of a function $\phi(x, y)$ ($\phi(x, y, z)$) is the intersection of the graph of the function with the plane $z = c$ ($w = c$). In what follows we will deal only with curves γ in $\mathbb{R}^2$ since surfaces σ in $\mathbb{R}^3$ can be treated similarly. Besides, we will learn that images are handled through a certain set of special curves.

Example The circle $x^2 + y^2 = 1$ can be regarded as the 0-level set ($\phi(x, y) = 0$) of the cone

$$\phi(x, y) = \sqrt{x^2 + y^2} - 1,$$

or the one-level set ($\phi(x, y) = 1$) of the cone

$$\phi(x, y) = \sqrt{x^2 + y^2}.$$

It is also the 0-level set of the semishpere

$$\phi(x, y) = \sqrt{3} - \sqrt{4 - x^2 - y^2}.$$

As the previous example indicates, there are many ways of realizing a given curve as the level set of a function but one immediate advantage of this point of view is that we are always dealing with functions rather than implicit equations representing curves or surfaces. Another advantage for dealing with functions is that it makes it easier to deal with level curves that change topology (split into two curves).

If $\gamma(t) = (x(t), y(t))$ is a parametrization of γ and if γ is the c-level curve of the function ϕ,then

$$\phi(x(t), y(t)) = c.$$

Remember that the gradient $\nabla\phi$ at any point $(x(t), y(t))$ on γ is orthogonal to γ. Indeed, if we differentiate the above equation with respect to t we get

$$\phi_x(x(t), y(t))\, x'(t) + \phi_y(x(t), y(t))\, y'(t) = 0,$$

where $(\cdot)'$ denotes differentiation with respect to the parameter t. This equation can be rewritten as

$$\langle \nabla\phi, \gamma' \rangle = 0. \tag{1}$$

Therefore, $\nabla\phi$ is in the direction of $\pm\overrightarrow{N}$.

When $\gamma = \gamma(t,\tau) = (x(t,\tau), y(t,\tau))$ is changing with respec to to time τ, we can still regard $\gamma(\cdot,\tau)$ as the level curve of a function $\phi(x,y,\tau)$ in a similar fashion. For example, for each τ, the circle $x^2 + y^2 = (r_0 - \tau)^2$ is the zero level curve of the time dependent function

$$\phi(x,y,\tau) = \sqrt{x^2 + y^2} - (r_0 - \tau).$$

The graph of this function for each fixed τ is the cone

$$z = \sqrt{x^2 + y^2} - (r_0 - \tau)$$

with vertex at $z = -(r_0 - \tau)$ which intersect the xy-plane in a circle of radius $(r_0 - \tau)$. Recall that the circles $x^2 + y^2 = (r_0 - \tau)^2$ are the evolution curves of the evolution equation

$$\begin{aligned} \frac{\partial\gamma}{\partial\tau} &= \overrightarrow{N}, \\ \gamma(t,0) &= \left\{(x,y) : x^2 + y^2 = r_0 - \tau\right\}. \end{aligned}$$

Observe that in the case of an evolving function ϕ, equation (1) still holds for each fixed τ: the gradient $\left(\phi_x(\cdot,\cdot,\tau), \phi_y(\cdot,\cdot,\tau)\right)$ of the function $\phi(\cdot,\cdot,\tau)$ is orthogonal to the level curve $\gamma(\cdot,\tau)$.

The question now is how do we infer the evolution equation governing the function ϕ from the evolution equation governing its level curves $\gamma(\cdot,\tau)$? For this we assume that γ is evolving according to the equation

$$\gamma_\tau = V\overrightarrow{N},$$

γ is positively oriented and that ϕ is chosen so that $\nabla\phi$ always points in the outward direction. In this case $\overrightarrow{N}$ (always pointing inward) is in the opposite direction of $\nabla\phi$. Since $\phi(x(\cdot,\tau), y(\cdot,\tau), \tau) = c$, differentiating with respect to τ gives

$$\begin{aligned} 0 &= \phi_x x_\tau + \phi_y y_\tau + \phi_\tau \\ &= \left\langle \nabla\phi, \frac{\partial\gamma}{\partial\tau} \right\rangle + \phi_\tau \\ &= \left\langle \nabla\phi, V\overrightarrow{N} \right\rangle + \phi_\tau \\ &= V\left\langle \nabla\phi, -\frac{\nabla\phi}{|\nabla\phi|} \right\rangle + \phi_\tau \\ &= -V|\nabla\phi| + \phi_\tau. \end{aligned}$$

Therefore, the evolution of ϕ is governed by

$$\phi_\tau = V|\nabla\phi|. \tag{2}$$

The above equation is also called the "*Eulerian Formulation*".

We can also get the geometric properties associated with the evolving curve γ from the function ϕ. For example, the following lemma shows how to compute the curvature.

Lemma 1 *The curvature κ of the level curve $\phi(\gamma,\tau) = c$ is given by*

$$\kappa = \frac{\phi_{xx}\phi_y^2 - 2\phi_{xy}\phi_x\phi_y + \phi_{yy}\phi_x^2}{|\nabla\phi|^3}.$$

Proof. Assume that γ is parametrized by arclength, positively oriented and ϕ is chosen such that $\overrightarrow{N} = -\frac{\nabla\phi}{|\nabla\phi|}$. Differentiating $\phi(\gamma,\tau) = c$ twice with respect to the space parameter we get

$$\begin{aligned} \langle \nabla\phi, \gamma' \rangle &= 0, \\ \left\langle \nabla^2\phi\gamma', \gamma' \right\rangle + \langle \nabla\phi, \gamma'' \rangle &= 0, \end{aligned}$$

where

$$\nabla^2\phi = \begin{bmatrix} \phi_{xx} & \phi_{xy} \\ \phi_{xy} & \phi_{yy} \end{bmatrix}.$$

Now

$$\begin{aligned} \gamma' &= R\left(-\frac{\pi}{2}\right)\overrightarrow{N} = -R\left(-\frac{\pi}{2}\right)\frac{\nabla\phi}{|\nabla\phi|} = \frac{1}{|\nabla\phi|}R\left(\frac{\pi}{2}\right)\nabla\phi \\ &= \frac{1}{|\nabla\phi|}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \phi_x \\ \phi_y \end{bmatrix} = \frac{1}{|\nabla\phi|}\begin{bmatrix} -\phi_y \\ \phi_x \end{bmatrix} \end{aligned}$$

and

$$\gamma'' = \kappa\overrightarrow{N} = -\frac{\kappa}{|\nabla\phi|}\nabla\phi.$$

Therefore,

$$\begin{aligned} \frac{1}{|\nabla\phi|^2}\left\langle \nabla^2\phi R\left(\frac{\pi}{2}\right)\nabla\phi, R\left(\frac{\pi}{2}\right)\nabla\phi \right\rangle - \frac{\kappa}{|\nabla\phi|}\langle \nabla\phi, \nabla\phi \rangle &= 0 \\ \frac{1}{|\nabla\phi|^2}\left\langle \nabla^2\phi R\left(\frac{\pi}{2}\right)\nabla\phi, R\left(\frac{\pi}{2}\right)\nabla\phi \right\rangle - \kappa|\nabla\phi| &= 0. \end{aligned}$$

Writing this equation in matrix form gives

$$\begin{aligned} \frac{1}{|\nabla\phi|^2}\begin{bmatrix} -\phi_y & \phi_x \end{bmatrix}\begin{bmatrix} \phi_{xx} & \phi_{xy} \\ \phi_{xy} & \phi_{yy} \end{bmatrix}\begin{bmatrix} -\phi_y \\ \phi_x \end{bmatrix} - \kappa|\nabla\phi| &= 0 \\ \frac{1}{|\nabla\phi|^2}\left(\phi_{xx}\phi_y^2 - 2\phi_{xy}\phi_x\phi_y + \phi_{yy}\phi_x^2\right) - \kappa|\nabla\phi| &= 0. \end{aligned}$$

Therefore,

$$\kappa = \frac{\phi_{xx}\phi_y^2 - 2\phi_{xy}\phi_x\phi_y + \phi_{yy}\phi_x^2}{|\nabla\phi|^3}.$$

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Observe that we can also write

$$\kappa = \nabla\cdot\left(\frac{\nabla\phi}{|\nabla\phi|}\right)$$

which can be verified by direct computation.

Example The Eulerian formulation (2) of the curvature flow

$$\frac{\partial\gamma}{\partial\tau} = \kappa\overrightarrow{N}$$

has the form

$$\begin{aligned} \frac{\partial\phi}{\partial\tau} &= \kappa|\nabla\phi| \\ &= \frac{\phi_{xx}\phi_y^2 - 2\phi_{xy}\phi_x\phi_y + \phi_{yy}\phi_x^2}{|\nabla\phi|^2} \\ &= \nabla\cdot\left(\frac{\nabla\phi}{|\nabla\phi|}\right)|\nabla\phi|. \end{aligned}$$

2 The Image Scale Space

An image is often perceived as a collection of shapes or elements where a hierarchical structure exists within these shapes. In other words complex images can be constructed from simpler elements. For example an image is composed of several shapes (segments). Each segment is composed of a hierarchy of minor detail, major details down to set of edges. This hierarchy of elements in an image is called the image *scale space.* The idea of image simplification, or image smoothing is to progressively decompose an image in an ascending scale of details. Smaller details (scales) are ignored as the scale increases. The idea is the same as drawing a map to increasing scales. At the small scales we are able to view more details on the map and so on. Moving from small scales to larger scales is an evolution process. This evolution process should not create new structures with the increasing scale. The simplest evolution equation that one can think of is the heat equation

$$I_\tau = \triangle I,$$

where $I(x, y, \tau)$ is the gray-level intensity function for the image at the point (x, y) and at time τ and $\triangle$ is the Laplace operator

$$\triangle u = u_{xx} + u_{yy}.$$

The initial condition $I(x, y, 0)$ is the image itself. This process simplifies images in the sense that small details are smoothed out with time but it does not preserve structures. A surface could change topology as the image evolves.

In an image, shapes are defined by their boundary curves. This view simplifies a two dimensional image into a collection of one dimensional curves. If we are to preserve the image properties as they are simplified we must have the following:

1. Shapes preserve their inclusion order when smoothed, without self intersecting.
2. All shapes smooth to circles then to points without developing singularities (such as self intersection).
3. The total curvature should be strictly decreasing unless the curve is a circle, in which case it remains constant.
4. The number of curvature extrema or inflection points is strictly decreasing.
5. This process should satisfy the semigroup property: evolving by amount $2t$ can be achieved by evolving by t in two sequential steps.

The above properties mean that shapes are preserved as they are simplified. This point of view, in turn, means that -with the above properties granted- we can evolve a gray level image by evolving its level set curves. Image smoothing is then obtained by superimposing the smoothed level sets. This reasoning now provides one more justification for considering curves as level sets of functions defined in higher dimension.

The question now is: What flow rules have the above five properties? We will learn in the coming chapters that curvature flows

$$\frac{\partial \phi}{\partial \tau} = \kappa \left| \nabla \phi \right|$$

satisfy all these properties.

3 Evolving Images as Surfaces

In the previous section we learned that we can evolve whole gray scale images by evolving its intensity level curves. This point of view has shortcomings when the image in question is colored instead of monochromatic. It is also ill suited to treat images defined on "bent" surfaces. To handle these difficulties we need to consider images as evolving surfaces instead of curves. In the coming chapters we will learn about the geometric framework for image processing in which images are considered as surfaces. We will also be interested in "bending invariant" evolutions. These flows should satisfy the following properties.

1. Invariance to surface bending.
2. Gray-level sets should be preserved during the flow.
3. The level sets should evolve into circles and then into points or converge to geodesics.
4. The image topology should be simplified.

All these properties are satisfied by geodesic curvature flows, in which curve on surfaces evolve according to the equation

$$\gamma_\tau = \kappa_g \mathbf{n}$$

where $\mathbf{n}$ is the principal normal given by

$$\mathbf{n} = \frac{1}{\kappa}\gamma_{ss}$$

and κ_g is the geodesic curvature obtained from the decomposition

$$\gamma_{ss} = \kappa_n \overrightarrow{N} + \kappa_g \overrightarrow{N} \times \gamma_s.$$

Such flows are implemented in the following manner: Consider a gray-level image (on a bent surface) and choose a specific level curve of this image. Project the flow equation of the curve onto the image coordinate plane and evolve using the Eulerian formulation.

HW: Implement a level set finder for a gray level image I. Write a function file that processes only four values $\{I_{ij}, I_{i+1,j}, I_{i+1,j+1}, I_{i,j+1}\}$ at a time and return a segment of the desired level set. Apply to a simple two level image and to the image of the clown.