KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
DEPARTMENT OF MATHEMATICAL SCIENCES
MATH 411
Final Part II
January, 2007
Take Home

1. **(5points)** Let $f : I \subset \mathbb{R}^1 \to \mathbb{R}^1$ be continuous. Show that the set
$$G = \{(x, f(x)) : x \in I\}$$
is connected.

2. **(5points)** Let f be a continuous function on the interval $[0,1]$. Define the set E by
$$E = \left\{ x \in [0,1] : f(x) = \max_{t \in [0,1]} f(t) \right\}.$$
Show that E is compact.

3. **(4points)** Show that if $\sum a_n$ is convergent and $a_n \geq 0$, then $\sum a_n^2$ converges. Give an example to show that the condition $a_n \geq 0$ cannot be dropped.

4. **(5points)** Find all extrema of the function
$$f(x, y) = \int_y^1 \frac{\sin xt}{t} dt$$
in the vertical strip
$$S = \left\{ (x, y) \in \mathbb{R}^2 : 0 < x < 10\pi. \right\}$$

5. **(5points)** Let F be a figure in $\mathbb{R}^N$ and suppose that $f : F \to \mathbb{R}^1$ is uniformly continuous. Show that f is Darboux integrable on F.

6. (a) **(4points)**Show that if $\{a_n\}$ is a bounded sequence and $b_n \to b_0$ then
$$\overline{\lim} a_n b_n = b_0 \overline{\lim} a_n.$$

   (b) **(4points)**Suppose $\{a_n\}$ is a sequence of positive numbers. Show that
$$\underline{\lim} \frac{a_{n+1}}{a_n} \leq \underline{\lim} \sqrt[n]{a_n} \leq \overline{\lim} \sqrt[n]{a_n} \leq \overline{\lim} \frac{a_{n+1}}{a_n}.$$

   (c) **(4points)**Show that
$$\overline{\lim} a_n = \lim_{n \to \infty} \sup_{k \geq n} \{a_k\}.$$

   (d) Compute $\overline{\lim} a_n$, $\underline{\lim} a_n$ for

      1. **(2points)** $a_n = \left(1 + \frac{1}{n}\right)\left(1 + \sin\left(\frac{n\pi}{8}\right)\right)^{1/n}$,
      2. **(2points)** $a_n = \frac{n + (-1)^n (2n+1)}{n}$.