Solutions of Exam 1.

1. y$' = \sqrt{x+y+1}$.

Let $v = x+y+1$. Then $v' = 1 + y'$. Substituting, we have

$$v' - 1 = \sqrt{v}$$

Separating the variables we get

$$\int \frac{dv}{1+\sqrt{v}} = \int dx$$

For the first integral, we use the substitution $u^2 = v, 2udu = dv$. It gives

$$\int \frac{2udu}{1+u} = x + C$$

By long division, we obtain

$$\begin{aligned} 2\int 1 - \frac{1}{1+u}du &= x + C \\ 2\left(u - \ln\left(1+u\right)\right) &= x + C \\ \sqrt{v} - \ln\left(1+\sqrt{v}\right) &= \frac{x}{2} + C \\ \sqrt{x+y+1} - \ln\left(1+\sqrt{x+y+1}\right) &= \frac{x}{2} + C. \end{aligned}$$

2. $xy' + 6y = 3xy^{4/3}$.

This is a Bernouli equation, we use the substitution $v = y^{1-4/3} = y^{-1/3}$. Then

$$\begin{aligned} y &= v^{-3} \\ y' &= -3v^{-4}v' \end{aligned}$$

The equation becomes

$$\begin{aligned} -3v^{-4}v' + \frac{6}{x}v^{-3} &= 3v^{-4} \\ v' - \frac{2}{x}v &= 3 \end{aligned}$$

which is a linear equation with integrating factor

$$e^{\int -\frac{2}{x}} = e^{-2\ln x} = \frac{1}{x^2}.$$

The equation becomes

$$\left(\frac{1}{x^2}v\right)' = \frac{3}{x^2}$$

Integrating, we get

$$\begin{aligned}\frac{1}{x^2}v &= -\frac{3}{x}+C\\ v &= -3x+Cx^2\\ y^{-1/3} &= .-3x+Cx^2\\ y &= \frac{1}{\left(-3x+Cx^2\right)^3}.\end{aligned}$$

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3. $(1-4xy^2)\,\frac{dy}{dx}=y^3.$

This equation is linear in x. So we can rewrite it in the form

$$\begin{aligned}\frac{1}{(1-4xy^2)}\frac{dx}{dy} &= \frac{1}{y^3}\\ \frac{dx}{dy} &= \frac{1}{y^3}-\frac{4x}{y}\\ \frac{dx}{dy}+\frac{4}{y}x &= \frac{1}{y^3}\end{aligned}$$

The integrating factor is

$$y^4$$

Multiplying by the integrating factor, we get

$$\left(y^4x\right)' = y$$

Solving,

$$\begin{aligned}y^4x &= \frac{1}{2}y^2+C\\ x &= \frac{1}{2y^2}+\frac{C}{y^4}.\end{aligned}$$

4. $yy'+x=\sqrt{x^2+y^2}.$

This is a homogenous equation. It can be written in the form

$$y'+\frac{x}{y}=\sqrt{1+\left(\frac{x}{y}\right)^2}.$$

Use the substitution

$$\begin{aligned}v &= \frac{y}{x}\\ vx &= y\\ xv'+v &= y'\end{aligned}$$

The equation becomes

$$
\begin{aligned}
xv' + v + \frac{1}{v} &= \sqrt{1 + \frac{1}{v^2}} \\
xv' + \frac{(v^2+1)}{v} &= \frac{\sqrt{v^2+1}}{v} \\
xv' &= \frac{\sqrt{v^2+1} - (v^2+1)}{v}
\end{aligned}
$$

Separating the variables give

$$
\int \frac{v}{\sqrt{v^2+1} - (v^2+1)} dv = \int \frac{1}{x} dx
$$

Using the substitution $u^2 = v^2 + 1,\ udu = vdv$ gives

$$
\begin{aligned}
\int \frac{udu}{u - u^2} &= \ln Cx \\
\int \frac{du}{1-u} &= \ln Cx \\
-\ln(u-1) &= \ln Cx \\
\frac{1}{\sqrt{v^2+1}} &= Cx \\
\frac{x}{\sqrt{x^2+y^2}} &= Cx \\
\sqrt{x^2+y^2} &= C.
\end{aligned}
$$

5. $y' + y\cot x = \cos x$.

   This is a linear equation with integrating facto $e^{\int \cot x} = e^{\ln \sin x} = \sin x$. Multiplying by the integrating factor gives

$$
\begin{aligned}
(\sin x\ y)' &= \cos x \sin x \\
\sin x\ y &= \int \cos x \sin x dx \\
\sin x\ y &= \frac{1}{2}\sin^2 x + C \\
y &= \frac{1}{2}\sin x + C\csc x.
\end{aligned}
$$

6. $yy'' = 3(y')^2$.

   Use the substitution $y' = v, y'' = v\frac{dv}{dy}$. This gives

$$
yv\frac{dv}{dy} = 3v^2
$$

Separate the variables to get

$$
\begin{aligned}
\int \frac{dv}{v} &= 3\int \frac{dy}{y} \\
v &= Cy^3 \\
y' &= Cy^3 \\
\int \frac{dy}{y^3} &= C\int dx \\
-\frac{1}{2y^2} &= Cx + D \\
y^2 &= \frac{1}{Cx + D} \\
y &= \frac{1}{\sqrt{Cx + D}}.
\end{aligned}
$$