KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
DEPARTMENT OF MATHEMATICAL SCIENCES
MATH 260
Exam # 3 (Solutions)

1. **(5points)** In the vector space $\mathbb{R}^4$ define the subsets

   $W_1 =$ the set of all vectors $v = (x_1, x_2, x_3, x_4)$ in $\mathbb{R}^4$ such that $x_1 + x_2 = x_3 - x_4$,

   $W_2 =$ the set of all vectors $v = (x_1, x_2, x_3, x_4)$ in $\mathbb{R}^4$ such that $x_1 x_2 = x_3 x_4$.

   Determine (giving reasons) whether or not W_1 and W_2 are subspaces of $\mathbb{R}^4$.

   Solution

   W_1 is a subspace of $\mathbb{R}^4$. To show this we need to show that W_1 is closed under vector addition and scalar multiplication. Let $v_1 = (x_1, x_2, x_3, x_4)$ and $v_2 = (y_1, y_2, y_3, y_4) \in W_1$. Then $x_1 + x_2 = x_3 - x_4$ and $y_1 + y_2 = y_3 - y_4$.

   $v_1 + v_2 = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4)$. Now

$$\begin{aligned}(x_1 + y_1) + (x_2 + y_2) &= (x_1 + x_2) + (y_1 + y_2) \\ &= (x_3 - x_4) + (y_3 - y_4) \\ &= (x_3 + y_3) - (x_4 + y_4).\end{aligned}$$

   Therefore, $v_1 + v_2 \in W_1$ and W_1 is closed under addition. Also, $cv_1 = (cx_1, cx_2, cx_3, cx_4)$ and

$$\begin{aligned}cx_{1+}cx_2 &= c(x_1 + x_2) \\ &= c(x_3 - x_4) \\ &= (cx_3 - cx_4).\end{aligned}$$

   Thus $cv_1 \in W_1$ and W_1 is closed under scalar multiplicatoin.

   W_2 is not a subspace for, if we take $v = (0, 2, 3, 0)$ and $w = (4, 0, 0, 5)$, then both $v, w \in W_2$, but $v + w = (4, 2, 3, 5) \notin W_2$. Thus W_2 is not closed under vector addition.

2. Is the vector $v = (-7, 7, 11)$ in the span of the vectors $v_1 = (1, 2, 1)$, $v_2 = (-4, -1, 7)$, $v_3 = (-3, 1, 3)$?. If so, express v as a linear combination of v_1, v_2, v_3.

   Solution

   To determine whether or not v is in the span of v_1, v_2, v_3, we try to solve the system

$$\begin{bmatrix} 1 & -4 & -3 \\ 2 & -1 & 1 \\ 1 & 7 & 3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 7 \\ 11 \end{bmatrix},$$

   for which the augmented matrix is

$$\begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 7 & 3 & 11 \end{bmatrix}.$$

The echlon form of the above matrix is

$$\begin{bmatrix} 1 & -4 & -3 & -7 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix},$$

which gives the solution $\alpha_1 = 3, \alpha_2 = 0, \alpha_3 = 2$. Therefore, $v = 3v_1 + 2v_3$.

3. If v_1, v_2, v_3 are linearly independent vectors in a vector space v, show that the vectors $u_1 = v_2 + v_3,\ u_2 = v_1 + v_3, u_3 = v_1 + v_2$ are also linearly independent.

Solution

Assume $\alpha_1 u_1 + \alpha_2 u_1 + \alpha_3 u_3 = 0$. Then

$$\begin{aligned} \alpha_1 (v_2 + v_3) + \alpha_2 (v_1 + v_3) + \alpha_3 (v_1 + v_2) &= 0 \\ (\alpha_2 + \alpha_3) v_1 + (\alpha_1 + \alpha_3) v_2 + (\alpha_1 + \alpha_2) v_3 &= 0. \end{aligned}$$

Since v_1, v_2, v_3 are linearly independent,

$$\begin{aligned} \alpha_2 + \alpha_3 &= 0, \\ \alpha_1 + \alpha_3 &= 0, \\ \alpha_1 + \alpha_2 &= 0. \end{aligned}$$

Solving this system, we get

$$\alpha_1 = \alpha_2 = \alpha_3 = 0.$$

Thus, u_1, u_2, u_3 are linearly independent.

4. Find a basis for the solution space of the homogeneous linear system

$$\begin{aligned} x_1 - 3x_2 - 9x_3 - 5x_4 &= 0 \\ 2x_1 + x_2 - 4x_3 + 11x_4 &= 0 \\ x_1 + 3x_2 + 3x_3 + 13x_4 &= 0 \end{aligned}$$

Solution

The coefficient matrix for this system is

$$\begin{bmatrix} 1 & -3 & -9 & -5 \\ 2 & 1 & -4 & 11 \\ 1 & 3 & 3 & 13 \end{bmatrix}$$

and its echlon form is

$$\begin{bmatrix} 1 & -3 & -9 & -5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, x_1, x_2 are leading variables and x_3, x_4 are free variables. Putting $x_3 = 1, x_4 = 0$, gives

$$u_1 = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

and putting $x_3 = 0, x_4 = 1$, gives

$$u_2 = \begin{bmatrix} -4 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

A basis vor the solution space is then $\{u1, u_2\}$.

5. Determine (with reasons) whether or not the functions $f(x) = x, g(x) = |x|$ are linearly independent on:

   (a) the real line,

   (b) the interval $I = (0, \infty)$.

   Solution

   (a) Assume $c_1 f(x) + c_2 g(x) = 0$, for all $x \in \mathbb{R}$. In particular, taking $x = -1, x = 1$, we get

$$\begin{aligned} -c_1 + c_2 &= 0, \\ c_1 + c_2 &= 0, \end{aligned}$$

   which implies that $c_1 = c_2 = 0$. Therefore, f, g are linearly independent on $\mathbb{R}$.

   (b), notice that for all $x \in I$, we have $f(x) = g(x) = x$. Therefore,

$$f(x) - g(x) = 0$$

   for all $x \in I$. Taking $c_1 = 1, c_2 = -1$, we get

$$c_1 f(x) + c_2 g(x) = 0$$

   for all $x \in I$. Therefore, f, g are linearly dependent on I.

6. Suppose $1, r, s$ are all distinct numbers. Show that$W(e^x, e^{rx}, e^{sx}) = e^{(1+r+s)x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & r & s \\ 1 & r^2 & s^2 \end{vmatrix}$ and hence, show that the functions e^x, e^{rx}, e^{sx} are linearlyindependent on the real line.

   Solution

$$\begin{aligned} W(e^x, e^{rx}, e^{sx}) &= \begin{vmatrix} e^x & e^{rx} & e^{sx} \\ e^x & re^{rx} & se^{sx} \\ e^x & r^2e^{rx} & s^2e^{sx} \end{vmatrix} = e^x \begin{vmatrix} 1 & e^{rx} & e^{sx} \\ 1 & re^{rx} & se^{sx} \\ 1 & r^2e^{rx} & s^2e^{sx} \end{vmatrix} \\ &= e^x e^{rx} \begin{vmatrix} 1 & 1 & e^{sx} \\ 1 & r & se^{sx} \\ 1 & r^2 & s^2e^{sx} \end{vmatrix} = e^x e^{rx} e^{sx} \begin{vmatrix} 1 & 1 & 1 \\ 1 & r & s \\ 1 & r^2 & s^2 \end{vmatrix} \\ &= e^{(1+r+s)x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & r & s \\ 1 & r^2 & s^2 \end{vmatrix}. \end{aligned}$$

To show linear independence we need to show that $W\left(e^{x}, e^{rx}, e^{sx}\right) \neq 0$ for all $x \in \mathbb{R}$.

$$\begin{aligned} W\left(e^{x}, e^{rx}, e^{sx}\right) &= e^{(1+r+s)x} \begin{vmatrix} 1 & 0 & 0 \\ 1 & r-1 & s-1 \\ 1 & r^{2}-1 & s^{2}-1 \end{vmatrix} \\ &= e^{(1+r+s)x} (r-1)(s-1) \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & r+1 & s+1 \end{vmatrix} \\ &= e^{(1+r+s)x} (r-1)(s-1)(s-r). \end{aligned}$$

Since the exponential function is never zero and the numbers $1, r, s$ are all distinct, $W\left(e^{x}, e^{rx}, e^{sx}\right) \neq 0$ for all $x \in \mathbb{R}$.