1 The vector space $\mathbb{R}^N$ and subspaces

A vector $\mathbf{v} \in \mathbb{R}^N$ is an n-tuple $\mathbf{v} = (v_1, v_2, \cdots, v_N)$. We will also write v as a column vector in the form

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix}.$$

1.1 Operations

We define two operations on vectors in $\mathbb{R}^N$: vector addition and scalar multiplication.

a) Vector Addition:

Let $\mathbf{v} = (v_1, v_2, \cdots, v_N)$, $\mathbf{w} = (w_1, w_2, \cdots, w_N)$ be two vectors in $\mathbb{R}^N$. We define

$$\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, \cdots, v_N + w_N).$$

b) Scalar Multiplication:

Let $\mathbf{v} = (v_1, v_2, \cdots, v_N) \in \mathbb{R}^N$ and $c \in \mathbb{R}$ be a scalar. The scalar multiplication of c and $\mathbf{v}$ is defined by

$$c\mathbf{v} = (cv_1, cv_2, \cdots, cv_N).$$

1.2 Properties

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^N, c, d \in \mathbb{R}$. We have the following properties:

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$. Here $\mathbf{0}$ is the zero vector in $\mathbb{R}^N$; $\mathbf{0} = (0, 0, \cdots, 0)$.
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{v} + c\mathbf{u}$.
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
7. $(cd)\mathbf{u} = c(d\mathbf{u}) = d(c\mathbf{u})$.

1.3 Length of a vector

The length of a vector $\mathbf{v} = (v_1, v_2, \cdots, v_N) \in \mathbb{R}^N$ is defined by

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + \cdots + v_N^2}.$$

1.4 Linear Dependence and Independence

The vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k \in \mathbb{R}^N$ are called linearly dependent if there exist scalars α_1,$\alpha_2, \cdots, \alpha_k \in \mathbb{R}$, not all zeros such that

$$\alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k = \mathbf{0}. \tag{1}$$

Otherwise $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are linearly independent.

Observe that equation (1) can be expressed as a linear system of N equations in the k unknowns α_1,$\alpha_2, \cdots, \alpha_k$. The vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are linearly independent if and only if the linear system (1) has only the trivial solution α_1=$\alpha_2 = \cdots = \alpha_k = 0$.

1.5 Examples

1. In $\mathbb{R}^3$ let $\mathbf{v}_1 = (2, 3, -1)\,, \mathbf{v}_2 = (2, 1, 1)\,, \mathbf{v}_3 = (4, 4, 0)\,.$ To decide if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent we try to find nontrivial solutions of the system

$$\alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0}. \tag{2}$$

We now write the above equation as a linear system of 3 equations in 3 unknowns.

$$\begin{aligned}\alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 &= \alpha_1\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + \alpha_2\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \alpha_3\begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2\alpha_1 + 2\alpha_2 + 4\alpha_3 \\ 3\alpha_1 + \alpha_2 + 4\alpha_3 \\ -\alpha_1 + \alpha_2 \end{bmatrix}\end{aligned}$$

We then want to solve the homogeneous system

$$\begin{aligned} 2\alpha_1 + 2\alpha_2 + 4\alpha_3 &= 0 \\ 3\alpha_1 + \alpha_2 + 4\alpha_3 &= 0 \\ -\alpha_1 + \alpha_2 \qquad &= 0 \end{aligned}$$

In matrix form we have

$$\begin{bmatrix} 2 & 2 & 4 \\ 3 & 1 & 4 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3}$$

The matrix of coefficients

$$\begin{bmatrix} 2 & 2 & 4 \\ 3 & 1 & 4 \\ -1 & 1 & 0 \end{bmatrix}$$

has the echlon form

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus we have α_3 as a free variable. This means that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent. To find nonzero values for $\alpha_1, \alpha_2, \alpha_3$ such that equation (2) is satisfied, we find all solutions of (3) first. For this we set $\alpha_3 = t$ and solve by back substitution to get $\alpha_2 = -t$ and $\alpha_1 = -t$. Therefore, for any value of t we have

$$-t\mathbf{v}_1 - t\mathbf{v}_2 + t\mathbf{v}_3 = \mathbf{0}.$$

Let us verify that.

$$-t\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} - t\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} -2t - 2t + 4t \\ -3t - t + 4t \\ t - t + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

as we expected.

Observe that we could also show that the system (2) has nontrivial solutions by checking the determinant of the matrix of coefficients

$$\begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

However, this works only if the matrix of coefficients is square so that we can compute the determinant. The more general approach so, is to reduce to echlon form.

2. As a second example, we take the vectors $\mathbf{v}_1 = (2, 3, -1, 1), \mathbf{v}_2 = (2, 1, 1, -2), \mathbf{v}_3 = (4, 4, 0, 5)$ in $\mathbb{R}^4$. We arrive at the system (3) simply by listing these vectors in columns. Thus we get

$$\begin{bmatrix} 2 & 2 & 4 \\ 3 & 1 & 4 \\ -1 & 1 & 0 \\ 1 & -2 & 5 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4}$$

The echlon form of the matrix of coefficients is

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since there are no free variables, the system (4) has only the trivial solution. Therefore $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent in $\mathbb{R}^4$.

We conclude from these two examples that in order to check for the linear independece of the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k \in \mathbb{R}^N$ we list them as columns in a matrix $A = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_k]$, reduce to echlon form (or compute the determinant if applicable) and see if we have free variables or not. More pricisely, if we let r be the number of nonzero rows in the echlon form of the matrix A, then the number of free varialbes n is given by

$$n = k - r.$$

Observe that, since A contains N rows, $r \leq N$. Consequenly, if $k > N$ then $n > 0$ and the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are linearly dependent in $\mathbb{R}^N$. For example, the vectors $\mathbf{v}_1 = (0,1), \mathbf{v}_2 = (2,-3), \mathbf{v}_3 = (1,2)$ are linearly dependent in $\mathbb{R}^2$ for in this case the homogeneous system is

$$\begin{bmatrix} 0 & 2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The echlon form of the matrix of coefficients is

$$\begin{bmatrix} 1 & -3 & 2 \\ 0 & 2 & 1 \end{bmatrix}.$$

There are 2 nonzero rows in this echlon form (corresponding to the leading variables α_1, α_2) and α_3 is free. Therefore, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent.

1.6 Subspaces of $\mathbb{R}^N$

A set V of vectors in $\mathbb{R}^N$ is called a subspace if it has the following two properties

a) If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$ (we say that V is closed under addition)

b) If $\mathbf{u} \in V$ and $c \in \mathbb{R}$ then $c\mathbf{u} \in V$ (we say that V is closed under scalar multiplication)

In words, we say that a subset V of $\mathbb{R}^N$ is a subspace if it is closed under vector addition and scalar multiplication. The two properties can be checked at once by showing that if $\mathbf{u}, \mathbf{v} \in V$ and $c \in \mathbb{R}$ then $\mathbf{u} + c\mathbf{v} \in V$. Clearly, $V = \mathbb{R}^N$ is a subspace. It is the maximal subspace of $\mathbb{R}^N$. Also, $V = \{\mathbf{0}\}$ is a subspace. It is the minimal subspace of $\mathbb{R}^N$. These two extreme cases are trivial cases.

1.7 Examples

1. Let $V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 = 0\}$. Let $\mathbf{u} = (u_1, 0, u_3), \mathbf{v} = (v_1, 0, v_3) \in V$ and $c \in \mathbb{R}$ then $\mathbf{u} + c\mathbf{v} = (u_1, 0, u_3) + c(v_1, 0, v_3) = (u_1 + cv_1, 0, u_3 + cv_3)$. Since the second component is zero, we see that $\mathbf{u} + c\mathbf{v} \in V$. Therefore, V is a subspace of $\mathbb{R}^3$.

2. $V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 = 1\}$. The two vectors $\mathbf{u} = (1,0,0,0)$ and $\mathbf{v} = (0,1,0,0)$ are in V, but $\mathbf{u}+\mathbf{v} = (1,1,0,0) \notin V$ since the sum of the first two components is not 1. Thus V is not closed under vector addition and, therefore, is not a subspace.

1.8 Solution Subspaces

Let A be an $M \times N$ matrix. We can regard any solution X of the homogeneous linear system

$$AX = \mathbf{0} \tag{5}$$

as a vector in $\mathbb{R}^N$. Let V be the set of solutions of the equation (5). We will show that V is a subspace of $\mathbb{R}^N$. Let $X, Y \in V$ (i.e., X, Y are solutions of (5)) and $c \in \mathbb{R}$. We need to show that $X + cY$ is a solution of (5). $A(X + cY) = AX + cAY = \mathbf{0} + c\mathbf{0} = \mathbf{0}$. Thus $X + cY \in V$.

1.9 Examples

1. Let $V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_4 = 0, x_1 - 2x_2 + 3x_3 - x_4 = 0\}$. Observe that the two conditions describing V can be written as a homogeneous system

$$\begin{aligned} x_1 \qquad\qquad\quad + x_4 &= 0 \\ x_1 - 2x_2 + 3x_3 - x_4 &= 0 \end{aligned}$$

   Or, in matrix form as

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & -2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6}$$

   Therefore, the vectors in V are solutions of the homogeneous system (6). Thus V is a subspace of $\mathbb{R}^4$.

2. The condition defining the set V in the second example in Section 1.7

$$\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 1$$

   is not homogeneous. That is why V is not a subspace of $\mathbb{R}^4$.

3. Let $V = \left\{(x_1, x_2, \cdots, x_N) \in \mathbb{R}^N : a_1x_1 + a_2x_4 + \cdots + a_Nx_N = 0\right\}$. The condition defining V can be written in the matrix form (5) where

$$A = [a_1 \; a_2 \; \cdots \; a_N]$$

   and

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}.$$

   It follows that V is a subspace of $\mathbb{R}^N$.

2 Linear Combinations and Span

A linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k \in \mathbb{R}^N$, is a vector $\mathbf{v}$ of the form

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k, \tag{7}$$

where c_1,$c_2, \cdots, c_k$ are scalars in $\mathbb{R}$. For example, a linear combination of the vectors $\mathbf{v}_1 = (2,3,-1,1)$, $\mathbf{v}_2 = (2,1,1,-2) \in \mathbb{R}^4$ is

$$\begin{aligned} \mathbf{v} &= 2\mathbf{v}_1 - 3\mathbf{v}_2 = 2(2,3,-1,1) - 3(2,1,1,-2) \\ &= (-2,3,-5,8). \end{aligned}$$

Observe that $\mathbf{v}$ is itself a vector in $\mathbb{R}^4$. Thus linear combination of vectors always produce a vector in the same space.

The first question we are interested in is: given a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k \in \mathbb{R}^N$ and another vector $\mathbf{v} \in \mathbb{R}^N$, is it possible to express $\mathbf{v}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$? In other words, can we find scalars c_1,$c_2, \cdots, c_k \in \mathbb{R}$ such that equation (7) holds? To answer this question, let $A = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_k]$ and $C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$. Then we can rewrite equation (7) in the form

$$AC = \mathbf{v},$$

which shows that the ansewer is in solving the above linear system for C. As usual, we write the augmented matrix $[A\ \mathbf{v}]$, reduce it to echlon form and determine the vector(s) C (if any) that satisfy the above linear system.

2.1 Examples

1. To determine whether the vector $\mathbf{v} = (2,-6,3)$ in $\mathbb{R}^3$ is a linear combination of the vectors $\mathbf{v}_1 = (1,-2,-1)$, $\mathbf{v}_2 = (1,-5,4)$, we write the linear system

$$\begin{bmatrix} 1 & 1 \\ -2 & -5 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}.$$

   The augmented matrix of this system is

$$\begin{bmatrix} 1 & 1 & 2 \\ -2 & -5 & -6 \\ -1 & 4 & 3 \end{bmatrix}.$$

   The reduced row echlon form is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   As you can see, the system is inconsistent. Therefore, $\mathbf{v}$ cannot be written as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$.

2. We repeat Example 1 for the vectors $\mathbf{v} = (-7,7,11)$, $\mathbf{v}_1 = (1,2,1)$, $\mathbf{v}_2 = (4,-1,2)$, $\mathbf{v}_3 = (-3,1,3)$. We have the system

$$\begin{bmatrix} 1 & 4 & -3 \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 7 \\ 11 \end{bmatrix}.$$

The augmented matrix is

$$\begin{bmatrix} 1 & 4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix}.$$

The reduced row echlon form is

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

This gives $c_1 = 2, c_2 = 0, c_3 = 3$. Thus $\mathbf{v} = 2\mathbf{v}_1 \ + 3\ \mathbf{v}_3 = 2\,(1,2,1) + 3\,(-3,1,3) = (-7,7,11)$.

2.2 The Span of a Set of Vectors

The set W of all linear combinations of of the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k \in \mathbb{R}^N$ is called the span of the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$. We use the notation

$$W = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}.$$

Theorem 1 *The span W of the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k \in \mathbb{R}^N$ is a subspace of $\mathbb{R}^N$.*

Proof. Let $\mathbf{u}, \mathbf{v} \ \in W$ and $c \in \mathbb{R}$. Then

$$\begin{aligned} \mathbf{u} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2, \cdots + c_k\mathbf{v}_k, \\ \mathbf{v} &= d_1\mathbf{v}_1 + d_2\mathbf{v}_2, \cdots + d_k\mathbf{v}_k. \\ \mathbf{u} + c\mathbf{v} &= (c_1 + cd_1)\,\mathbf{v}_1 + (c_2 + cd_2)\,\mathbf{v}_2 + \cdots + (c_k + cd_k)\,\mathbf{v}_k \\ &= e_1\mathbf{v}_1 + e_2\mathbf{v}_2, \cdots + e_k\mathbf{v}_k, \end{aligned}$$

where $e_1 = (c_1 + cd_1), e_2 = (c_2 + cd_2), \cdots, e_k = (c_k + cd_k)$. Therefore, $\mathbf{u} + c\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ and, hence, $\mathbf{u} + c\mathbf{v} \in W$. ■

The question raised in the previous section about whether a given $\mathbf{v}$ can be written as a linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ can be rephrased by asking whether a given vector $\mathbf{v}$ is in the span of the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$.

2.3 Linear Independence in $\mathbb{R}^N$

Suppose we are given the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k \in \mathbb{R}^N$ and we know that they are linearly independent. Let $A = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_k]$, $r = \#$ of nonzero rows in the echlon form of A and $n = k - r = \#$ of free variables in echlon form of A. Since $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are linealy independent, $n = 0$. Since $r \leq N$, we must also have $k \leq N$. If $k = N$, then $n = 0$ gives $r = N$ and there are no zero rows in the echlon form of A.If $k < N$, then $n = 0$ gives $r = k$ and the last $N - k$ rows of the echlon form of A are all zeros.