Solving Linear Systems Without Computing Eigenvectors

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The following method is used to solve linear systems of the form

$$X' = AX \tag{1}$$

where A is an $n \times n$ constant matrix and X is an $n \times 1$ vector.

1 The Method

The method consists of the following 4 steps.

Step 1: List the eigenvalues

$$\lambda_1, \lambda_2, \ldots, \lambda_n$$

of the matrix A, each one repeated as many times as its multiplicity.

Step 2: Compute the matrices $P_0, P_1, \ldots, P_{n-1}$ from the recursions

$$\begin{aligned} P_0 &= I, \\ P_k &= (A - \lambda_k I) P_{k-1}, \; k = 1, 2, \ldots, n-1. \end{aligned}$$

Step 3: Comute the functions $r_0(t), r_1(t), \ldots, r_{n-1}(t)$ from the recursions

$$\begin{aligned} r_0(t) &= e^{\lambda_1 t}, \\ r_k(t) &= e^{\lambda_{k+1} t} \int_0^t e^{-\lambda_{k+1} s} r_{k-1}(s)\, ds, \; k = 1, 2, \ldots, n-1. \end{aligned} \tag{2}$$

Step 4: Form the matrix

$$F(t) = \sum_{k=0}^{n-1} r_k(t) P_k.$$

The columns of F give a linearly independent set of solutions of (1). This means that the general solution of (1) is

$$X(t) = F(t) C$$

where C is a vector of arbitrary constants.

Remarks

1. The integration in equation (2) can be taken from any other initial point t_0 instead of 0.

2. The following formula, which is obtained from integration by parts is useful

$$\int t^m e^{at} dt = e^{at} \sum_{k=0}^{m} (-1)^k \frac{m!}{(m-k)!} \frac{t^{m-k}}{a^{k+1}}.$$

2 Examples

1. Solve the initial value problem

$$X' = \begin{bmatrix} -5 & 3 & -1 \\ 1 & 2 & 1 \\ 42 & -17 & 10 \end{bmatrix} X, \quad X(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

The eigenvalues for this problem are

$$\lambda_1 = 2, \lambda_2 = 2, \lambda_3 = 3.$$

The matrices P_0, P_1, P_2 are

$$\begin{aligned} P_0 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ P_1 &= A - 2I = \begin{bmatrix} -7 & 3 & -1 \\ 1 & 0 & 1 \\ 42 & -17 & 8 \end{bmatrix}, \\ P_2 &= (A - 2I) P_1 = \begin{bmatrix} -7 & 3 & -1 \\ 1 & 0 & 1 \\ 42 & -17 & 8 \end{bmatrix}^2 = \begin{bmatrix} 10 & -4 & 2 \\ 35 & -14 & 7 \\ 25 & -10 & 5 \end{bmatrix}. \end{aligned}$$

The functions $r_0(t), r_1(t), r_2(t)$ are

$$\begin{aligned}
r_0(t) &= e^{2t}, \\
r_1(t) &= e^{2t}\int_0^t e^{-2s} r_0(s)\, ds \\
&= e^{2t}\int_0^t e^{-2s} e^{2s} ds = te^{2t}, \\
r_2(t) &= e^{3t}\int_0^t e^{-3s} r_1(s)\, ds \\
&= e^{3t}\int_0^t e^{-3s} s e^{2s} ds \\
&= e^{3t}\int_0^t s e^{-s} ds \\
&= e^{3t}\left(1 - te^{-t} - e^{-t}\right) \\
&= e^{3t} - (1+t)\, e^{2t}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
F(t) &= e^{2t}P_0 + te^{2t}P_1 + \left(e^{3t} - (1+t)\, e^{2t}\right) P_2 \\
&= e^{2t}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + te^{2t}\begin{bmatrix} -7 & 3 & -1 \\ 1 & 0 & 1 \\ 42 & -17 & 8 \end{bmatrix} + \left(e^{3t} - (1+t)\, e^{2t}\right)\begin{bmatrix} 10 & -4 & 2 \\ 35 & -14 & 7 \\ 25 & -10 & 5 \end{bmatrix} \\
&= \begin{bmatrix} -9e^{2t} - 17te^{2t} + 10e^{3t} & 4e^{2t} + 7te^{2t} - 4e^{3t} & -2e^{2t} - 3te^{2t} + 2e^{3t} \\ -35e^{2t} - 34te^{2t} + 35e^{3t} & 15e^{2t} + 14te^{2t} - 14e^{3t} & -7e^{2t} - 6te^{2t} + 7e^{3t} \\ -25e^{2t} + 17te^{2t} + 25e^{3t} & 10e^{2t} - 7te^{2t} - 10e^{3t} & -4e^{2t} + 3te^{2t} + 5e^{3t} \end{bmatrix}.
\end{aligned}$$

The general Solution is

$$X(t) = F(t)\, C$$

Substituting $t = 0$ and $X(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ gives

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} C = C.$$

Therefore, the solution satisfying the initial condition is

$$\begin{aligned}
X(t) &= F(t)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} -7e^{2t} - 12te^{2t} + 8e^{3t} \\ -26e^{2t} - 24te^{2t} + 28e^{3t} \\ -17e^{2t} + 12te^{2t} + 20e^{3t} \end{bmatrix}.
\end{aligned}$$

2. Solve the system

$$X' = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} X.$$

The eigenvalues are

$$\lambda_1 = i, \lambda_2 = -i.$$

The matrices P_0, P_1 are

$$P_0 = I, P_1 = \begin{bmatrix} 1-i & 2 \\ -1 & -1-i \end{bmatrix}.$$

The functions r_0, r_1 are

$$\begin{aligned} r_0 &= e^{it}, \\ r_1 &= e^{-it} \int_0^t e^{is} e^{is} ds = e^{-it} \int_0^t e^{2is} ds \\ &= e^{-it} \frac{e^{2it} - 1}{2i} = \frac{e^{it} - e^{-it}}{2i} = \sin t. \end{aligned}$$

The function F is

$$\begin{aligned} F &= e^{it} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin t \begin{bmatrix} 1-i & 2 \\ -1 & -1-i \end{bmatrix} \\ &= (\cos t + i \sin t) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (\sin t) \begin{bmatrix} 1-i & 2 \\ -1 & -1-i \end{bmatrix} \\ &= \begin{bmatrix} \cos t & 0 \\ 0 & \cos t \end{bmatrix} + i \begin{bmatrix} \sin t & 0 \\ 0 & \sin t \end{bmatrix} + \begin{bmatrix} \sin t & 2\sin t \\ -\sin t & -\sin t \end{bmatrix} - i \begin{bmatrix} \sin t & 0 \\ 0 & \sin t \end{bmatrix} \\ &= \begin{bmatrix} \cos t + \sin t & 2\sin t \\ -\sin t & \cos t - \sin t \end{bmatrix}. \end{aligned}$$

The general solution is

$$\begin{bmatrix} \cos t + \sin t & 2\sin t \\ -\sin t & \cos t - \sin t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \cos t + c_1 \sin t + 2c_2 \sin t \\ c_2 \cos t - c_1 \sin t - c_2 \sin t \end{bmatrix}.$$

3. Solve the system

$$X' = \begin{bmatrix} 3 & -4 & 1 & 0 \\ 4 & 3 & 0 & 1 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{bmatrix} X$$

Here the eigenvalues are complex and repeated.

$$\lambda_1 = 3+4i,\ \lambda_2 = 3+4i,\ \lambda_3 = 3-4i,\ \lambda_4 = 3-4i$$

The matrices $P_0,\ P_1,\ P_2\ ,\ P_3$ are

$$P_0 = I,\ P_1 = \begin{bmatrix} -4i & -4 & 1 & 0 \\ 4 & -4i & 0 & 1 \\ 0 & 0 & -4i & -4 \\ 0 & 0 & 4 & -4i \end{bmatrix},$$

$$P_2 = \begin{bmatrix} -32 & 32i & -8i & -8 \\ -32i & -32 & 8 & -8i \\ 0 & 0 & -32 & 32i \\ 0 & 0 & -32i & -32 \end{bmatrix},\ P_3 = \begin{bmatrix} 0 & 0 & -32 & 32i \\ 0 & 0 & -32i & -32 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and the functions r_0, r_1, r_2, r_3 are $r_0 = e^{(3+4i)t}, r_1 = te^{(3+4i)t}, r_2 = -\frac{i}{8}\left(te^{(3+4i)t} - \frac{1}{4}e^{3t}\sin 4t\right), r_3 = -\frac{1}{32}e^{3t}\left(t\cos 4t - \frac{1}{4}\sin 4t\right)$.

$$\begin{aligned} F &= e^{(3+4i)t}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + te^{(3+4i)t}\begin{bmatrix} -4i & -4 & 1 & 0 \\ 4 & -4i & 0 & 1 \\ 0 & 0 & -4i & -4 \\ 0 & 0 & 4 & -4i \end{bmatrix} \\ &\quad -\frac{i}{8}\left(te^{(3+4i)t} - \frac{1}{4}e^{3t}\sin 4t\right)\begin{bmatrix} -32 & 32i & -8i & -8 \\ -32i & -32 & 8 & -8i \\ 0 & 0 & -32 & 32i \\ 0 & 0 & -32i & -32 \end{bmatrix} \\ &\quad -\frac{1}{32}e^{3t}\left(t\cos 4t - \frac{1}{4}\sin 4t\right)\begin{bmatrix} 0 & 0 & -32 & 32i \\ 0 & 0 & -32i & -32 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} e^{3t}\cos 4t & -e^{3t}\sin 4t & t\cos 4te^{3t} & -te^{3t}\sin 4t \\ e^{3t}\sin 4t & e^{3t}\cos 4t & te^{3t}\sin 4t & t\cos 4te^{3t} \\ 0 & 0 & e^{3t}\cos 4t & -e^{3t}\sin 4t \\ 0 & 0 & e^{3t}\sin 4t & e^{3t}\cos 4t \end{bmatrix}. \end{aligned}$$

Obseve that the matrix function $F(t)$ is always real, whether or not the eigenvalues are real.

4. Solve the system

$$X' = \begin{bmatrix} 11 & -1 & 26 & 6 & -3 \\ 0 & 3 & 0 & 0 & 0 \\ -9 & 0 & -24 & -6 & 3 \\ 3 & 0 & 9 & 5 & -1 \\ -48 & -3 & -138 & -30 & 18 \end{bmatrix} X$$

This system has eigenvalues

$$\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 2, \lambda_4 = 3, \lambda_5 = 3$$

The eigen value $\lambda = 2$ corresponds to two eigenvectors

$$v_1 = \begin{bmatrix} 8 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$

and the egenvector $\lambda = 3$ corresponds to three eigenvectors

$$v_3 = \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 3 \end{bmatrix}, v_5 = \begin{bmatrix} -2 \\ 2 \\ 0 \\ 3 \\ 0 \end{bmatrix}.$$

Thus the system has a set of 5 linearly independent solutions and the general solution is

$$\begin{aligned} X(t) &= e^{2t}(c_1 \begin{bmatrix} 8 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}) + e^{3t}(c_3 \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 3 \end{bmatrix} + c_5 \begin{bmatrix} -2 \\ 2 \\ 0 \\ 3 \\ 0 \end{bmatrix}) \\ &= \begin{bmatrix} 8c_1e^{2t} + c_2e^{2t} - 3c_3e^{3t} + c_4e^{3t} - 2c_5e^{3t} \\ 2c_3e^{3t} - c_4e^{3t} + 2c_5e^{3t} \\ -3c_1e^{2t} + c_3e^{3t} \\ c_1e^{2t} + 3c_5e^{3t} \\ 3c_2e^{2t} + 3c_4e^{3t} \end{bmatrix}. \end{aligned}$$

On the other hand, working with the above method, we have

$$\begin{aligned} P_0 &= I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, P_1 = \begin{bmatrix} 9 & -1 & 26 & 6 & -3 \\ 0 & 1 & 0 & 0 & 0 \\ -9 & 0 & -26 & -6 & 3 \\ 3 & 0 & 9 & 3 & -1 \\ -48 & -3 & -138 & -30 & 16 \end{bmatrix} \\ P_2 &= P_3 = P_4 = 0 \end{aligned}$$

we thus need to compute only $r_0, r_1, .$

$$r_0 = e^{2t}, r_1 = e^{3t} - e^{2t}$$

Thus

$$F = e^{2t}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \left(e^{3t} - e^{2t}\right)\begin{bmatrix} 9 & -1 & 26 & 6 & -3 \\ 0 & 1 & 0 & 0 & 0 \\ -9 & 0 & -26 & -6 & 3 \\ 3 & 0 & 9 & 3 & -1 \\ -48 & -3 & -138 & -30 & 16 \end{bmatrix}$$

$$= e^{2t}\begin{bmatrix} -8 & 1 & -26 & -6 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 9 & 0 & 27 & 6 & -3 \\ -3 & 0 & -9 & -2 & 1 \\ 48 & 3 & 138 & 30 & -15 \end{bmatrix} + e^{3t}\begin{bmatrix} 9 & -1 & 26 & 6 & -3 \\ 0 & 1 & 0 & 0 & 0 \\ -9 & 0 & -26 & -6 & 3 \\ 3 & 0 & 9 & 3 & -1 \\ -48 & -3 & -138 & -30 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} -8e^{2t}+9e^{3t} & e^{2t}-e^{3t} & -26e^{2t}+26e^{3t} & -6e^{2t}+6e^{3t} & 3e^{2t}-3e^{3t} \\ 0 & e^{3t} & 0 & 0 & 0 \\ 9e^{2t}-9e^{3t} & 0 & 27e^{2t}-26e^{3t} & 6e^{2t}-6e^{3t} & -3e^{2t}+3e^{3t} \\ -3e^{2t}+3e^{3t} & 0 & -9e^{2t}+9e^{3t} & -2e^{2t}+3e^{3t} & e^{2t}-e^{3t} \\ 48e^{2t}-48e^{3t} & 3e^{2t}-3e^{3t} & 138e^{2t}-138e^{3t} & 30e^{2t}-30e^{3t} & -15e^{2t}+16e^{3t} \end{bmatrix}$$

The general solution is

$$\begin{bmatrix} -8e^{2t}+9e^{3t} & e^{2t}-e^{3t} & -26e^{2t}+26e^{3t} & -6e^{2t}+6e^{3t} & 3e^{2t}-3e^{3t} \\ 0 & e^{3t} & 0 & 0 & 0 \\ 9e^{2t}-9e^{3t} & 0 & 27e^{2t}-26e^{3t} & 6e^{2t}-6e^{3t} & -3e^{2t}+3e^{3t} \\ -3e^{2t}+3e^{3t} & 0 & -9e^{2t}+9e^{3t} & -2e^{2t}+3e^{3t} & e^{2t}-e^{3t} \\ 48e^{2t}-48e^{3t} & 3e^{2t}-3e^{3t} & 138e^{2t}-138e^{3t} & 30e^{2t}-30e^{3t} & -15e^{2t}+16e^{3t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix}$$

5. Solve the system

$$X' = \begin{bmatrix} -4 & -5 & -3 & 1 & -2 & 0 & 1 & -2 \\ 4 & 7 & 3 & -1 & 3 & 0 & -1 & 2 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 2 & -4 & 2 & 0 & -3 & 1 \\ -8 & -14 & -5 & 1 & -6 & 0 & 1 & -4 \\ 4 & 7 & 4 & -3 & 3 & -1 & -3 & 4 \\ 2 & -2 & -2 & 5 & -3 & 0 & 4 & -1 \\ 6 & 7 & 3 & 0 & 2 & 0 & 0 & 3 \end{bmatrix} X.$$

In this case the eigenvalues are

$$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 1, \lambda_5 = 1, \lambda_6 = -1, \lambda_7 = -1, \lambda_8 = -1.$$

The matrices $P_0 \ldots P_7$ are

$$
\begin{aligned}
P_0 &= I, \\
P_1 &= \begin{bmatrix} -4 & -5 & -3 & 1 & -2 & 0 & 1 & -2 \\ 4 & 7 & 3 & -1 & 3 & 0 & -1 & 2 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 2 & -4 & 2 & 0 & -3 & 1 \\ -8 & -14 & -5 & 1 & -6 & 0 & 1 & -4 \\ 4 & 7 & 4 & -3 & 3 & -1 & -3 & 4 \\ 2 & -2 & -2 & 5 & -3 & 0 & 4 & -1 \\ 6 & 7 & 3 & 0 & 2 & 0 & 0 & 3 \end{bmatrix} \\
P_2 &= \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & -1 & 0 \\ -4 & -7 & -3 & 1 & -3 & 0 & 1 & -2 \\ -4 & -9 & -3 & 1 & -4 & 0 & 0 & -2 \\ 1 & 2 & 0 & 1 & 1 & 0 & 1 & 0 \\ 5 & 7 & 2 & 0 & 3 & 1 & 0 & 2 \\ 5 & 10 & 2 & 1 & 4 & 0 & 2 & 2 \\ 6 & 9 & 2 & 1 & 3 & 0 & 1 & 3 \end{bmatrix}, \\
P_3 &= \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 3 & 1 & -3 & 3 & 0 & -2 & 1 \\ -3 & -6 & -2 & 1 & -3 & 0 & 1 & -2 \\ 0 & 1 & 1 & -2 & 0 & -1 & -2 & 2 \\ 3 & -2 & -1 & 5 & -4 & 0 & 4 & -1 \\ 7 & 9 & 3 & 1 & 2 & 0 & 1 & 3 \end{bmatrix} \\
&,
\end{aligned}
$$

$$P_4 = \begin{bmatrix} -4 & -8 & -2 & 0 & -4 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -12 & -3 & 4 & -8 & 0 & 2 & -3 \\ 4 & 8 & 2 & 0 & 4 & 0 & 0 & 2 \\ 10 & 14 & 4 & 2 & 6 & 2 & 2 & 2 \\ 6 & 20 & 5 & -4 & 12 & 0 & -2 & 5 \\ 4 & 8 & 2 & 0 & 4 & 0 & 0 & 2 \end{bmatrix},$$

$$P_5 = \begin{bmatrix} 8 & 16 & 4 & 0 & 8 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 32 & 8 & -8 & 20 & 0 & -4 & 8 \\ -8 & -16 & -4 & 0 & -8 & 0 & 0 & -4 \\ -20 & -28 & -8 & -4 & -12 & -4 & -4 & -4 \\ -16 & -48 & -12 & 8 & -28 & 0 & 4 & -12 \\ -8 & -16 & -4 & 0 & -8 & 0 & 0 & -4 \end{bmatrix},$$

$$P_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8 & -16 & -4 & 0 & -8 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 16 & 4 & 0 & 8 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, P_7 = 0$$

The functions $r_0 \ldots r_6$ are

$$r_0 = 1, r_1 = t, r_2 = \frac{1}{2}t^2, r_3 = -\left(1 + t + \frac{1}{2}t^2\right) + e^t, r_4 = \left(3 + 2t + \frac{1}{2}t^2\right) + (t-3)\,e^t,$$

$$r_5 = \left(2 + t + \frac{1}{2}t^2\right) + \left(\frac{1}{2}t - \frac{7}{4}\right)e^t - \frac{1}{4}e^{-t}, r_6 = 2 + \frac{1}{2}t^2 + \left(\frac{1}{4}t - 1\right)e^t - \left(\frac{1}{4}t + 1\right)e^{-t}$$

$$
\begin{aligned}
F &= P_0 + tP_1 + \frac{1}{2}t^2 P_2 + \left(- \left(1 + t + \frac{1}{2}t^2 \right) + e^t \right) P_3 \\
&\quad + \left(\left(3 + 2t + \frac{1}{2}t^2 \right) + (t-3)\, e^t \right) P_4 + \left(\left(2 + t + \frac{1}{2}t^2 \right) + \left(\frac{1}{2}t - \frac{7}{4} \right) e^t - \frac{1}{4}e^{-t} \right) P_5 \\
&\quad + \left(2 + \frac{1}{2}t^2 + \left(\frac{1}{4}t - 1 \right) e^t - \left(\frac{1}{4}t + 1 \right) e^{-t} \right) P_6 \\
&= (P_0 - P_3 + 3P_4 + 2P_5 + 2P_6) \\
&\quad + t\,(P_1 - P_3 + 2P_4 + P_5) \\
&\quad + \frac{1}{2}t^2\,(P_2 - P_3 + P_4 + P_5 + P_6) \\
&\quad + e^t \left(P_3 - 3P_4 - \frac{7}{4}P_5 - P_6 \right) \\
&\quad + te^t \left(P_4 + \frac{1}{2}P_5 + \frac{1}{4}P_6 \right) \\
&\quad + e^{-t} \left(-\frac{1}{4}P_5 - P_6 \right) \\
&\quad + te^{-t} \left(-\frac{1}{4} \right) P_6.
\end{aligned}
$$

$$
= \begin{bmatrix} 5 & 7 & 2 & 0 & 3 & 0 & 0 & 2 \\ 1 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 1 & -1 & 0 & 0 & -1 & 0 \\ -5 & -7 & -2 & 0 & -3 & 0 & 0 & -2 \\ -1 & -2 & 0 & -1 & 0 & 0 & -1 & 0 \\ -10 & -15 & -5 & 0 & -6 & 0 & 0 & -4 \\ -1 & -2 & 0 & -1 & 0 & 0 & -1 & 0 \\ -11 & -17 & -5 & -1 & -6 & 0 & -1 & -4 \end{bmatrix}
+ t \begin{bmatrix} -4 & -6 & -3 & 1 & -3 & 0 & 1 & -2 \\ 5 & 8 & 3 & 0 & 3 & 0 & 0 & 2 \\ -1 & -2 & 0 & -1 & 0 & 0 & -1 & 0 \\ 4 & 6 & 3 & -1 & 3 & 0 & -1 & 2 \\ -5 & -8 & -3 & 0 & -3 & 0 & 0 & -2 \\ 4 & 6 & 3 & -1 & 3 & 0 & -1 & 2 \\ -5 & -8 & -3 & 0 & -3 & 0 & 0 & -2 \\ -1 & -2 & 0 & -1 & 0 & 0 & -1 & 0 \end{bmatrix}
$$

$$
+ \frac{1}{2}t^2 \begin{bmatrix} 5 & 8 & 3 & 0 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -5 & -8 & -3 & 0 & -3 & 0 & 0 & -2 \\ -5 & -8 & -3 & 0 & -3 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -5 & -8 & -3 & 0 & -3 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -5 & -8 & -3 & 0 & -3 & 0 & 0 & -2 \end{bmatrix}
+ e^t \begin{bmatrix} -2 & -3 & -1 & 0 & -1 & 0 & 0 & -1 \\ -1 & -1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & -1 & 0 \\ -1 & -2 & -1 & 1 & -1 & 0 & 1 & -1 \\ 5 & 8 & 3 & -1 & 3 & 0 & -1 & 3 \\ 5 & 6 & 1 & 3 & 1 & 0 & 3 & 1 \\ 9 & 13 & 4 & 1 & 4 & 0 & 1 & 4 \end{bmatrix}
$$

$$
+ e^{-t} \begin{bmatrix} -2 & -4 & -1 & 0 & -2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 8 & 2 & 2 & 3 & 0 & 1 & 2 \\ 2 & 4 & 1 & 0 & 2 & 0 & 0 & 1 \\ 5 & 7 & 2 & 1 & 3 & 1 & 1 & 1 \\ -4 & -4 & -1 & -2 & -1 & 0 & -1 & -1 \\ 2 & 4 & 1 & 0 & 2 & 0 & 0 & 1 \end{bmatrix}
+ te^{-t} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 4 & 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -4 & -1 & 0 & -2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

3 Motivation

The algorithm of the previous section is based on the following reasonings. First, observe that the general solution of the system

$$X' = AX \tag{3}$$

is

$$X = e^{tA}C$$

where C is an arbitraty constant vector and the matrix e^{tA} is evaluated from the infinite series

$$e^{tA} = \sum_{n=0}^{\infty} \frac{A^n}{n!} t^n.$$

We do not actually compute an infinite series here. Thanks to the Cayley Hamilto theorem, if $p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_0$ is the characteristic polynomial for A, then $p(A) = 0$. This means that we can write A^n in terms of lower powers of A as

$$A^n = -\left(a_{n-1}A^{n-1} + a_{n-2}A^{n-2} \ldots + a_0 I\right)$$

Similarly, we can use the above equation to write $A^m, m > n$ in terms of $A^{n-1}, A^{n-2} \dots, I$. A slight modification of the above idea asserts that we can write e^{tA} in terms of the matrices $P_0, P_1, \dots P_{n-1}$ defined above. The advantaged here is that $P_0, P_1, \dots P_{n-1}$ satisfy the recursion above, which is easy to compute with. Thus if we write

$$e^{tA} = \sum_{k=0}^{n-1} r_k(t) P_k, \tag{4}$$

then by substituting in (3), we get

$$\sum_{k=0}^{n-1} r'_k(t) P_k = \sum_{k=0}^{n-1} r_k(t) AP_k.$$

From the recursion, we have

$$\begin{aligned}
\sum_{k=0}^{n-1} r'_k(t) P_k &= \sum_{k=0}^{n-1} r_k(t) (P_{k+1} + \lambda_{k+1} P_k) \\
&= \sum_{k=0}^{n-1} r_k(t) P_{k+1} + \sum_{k=0}^{n-1} \lambda_{k+1} r_k(t) P_k \\
&= \sum_{k=1}^{n-1} r_{k-1}(t) P_k + \sum_{k=0}^{n-1} \lambda_{k+1} r_k(t) P_k \\
&= \lambda_1 r_0(t) I + \sum_{k=1}^{n-1} (\lambda_{k+1} r_k(t) + r_{k-1}(t)) P_k.
\end{aligned}$$

Thus, the functions $r_0(t), r_1(t), \dots, r_{n-1}(t)$ satisfy the differential equatoins

$$\begin{aligned}
r'_0(t) &= \lambda_1 r_0(t), \\
r'_k(t) &= \lambda_{k+1} r_k(t) + r_{k-1}(t), \ k = 1, 2, \dots, n-1.
\end{aligned}$$

To decide on the initial conditions, observe that, from (4) we have

$$e^{0A} = I = \sum_{k=0}^{n-1} r_k(0) P_k,$$

which suggests that we set $r_0(0) = 1, r_0(0) = r_1(0) = \dots = 0$. Solving the recursive first order differential equations we obtain

$$\begin{aligned}
r_0(t) &= e^{\lambda_1 t}, \\
r_k(t) &= e^{\lambda_{k+1} t} \int_0^t e^{-\lambda_{k+1} s} r_{k-1}(s) \, ds, \ k = 1, 2, \dots, n-1.
\end{aligned}$$