KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
DEPARTMENT OF MATHEMATICAL SCIENCES
MATH 260
Final Exam, Fall 061
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1. Find a general solution of the differential equations

   (a) $e^x + ye^{xy} + (e^y + xe^{xy})\, y' = 0$
   Rewrite the equation in the form

$$\left(e^x + ye^{xy}\right) dx + \left(e^y + xe^{xy}\right) dy = 0$$

   The equation is exact. Integrating the coefficient of dx *with respect to* x *gives*

$$F(x, y) = e^x + e^{xy} + h(y).$$

   Now

$$F_y = xe^{xy} + h'(y) = e^y + xe^{xy}.$$

   Thus

$$h'(y) = e^y$$

   and

$$h(y) = e^y.$$

   The solution is

$$e^x + e^{xy} + e^y = C.$$

   (b) $xy' + 2y = 6x^2\sqrt{y}$.
   This is a Bernouli equation with $m = 1/2$. *Therefore, we use the substitnution* $u = y^{1/2}$. *The equation becomes*

$$u' + \frac{1}{x}u = 3x.$$

   The integrating factor is

$$p(x) = e^{\int \frac{1}{x}} = x$$

   and we have

$$(xu)' = 3x^2.$$

   Then

$$xu = x^3 + C.$$

   In terms of y *we get*

$$y^{1/2} = x^2 + \frac{C}{x}$$

   or

$$y = \left(x^2 + \frac{C}{x}\right)^2.$$

2. (a) If $y_c = x^2(c_1 + c_2 \ln x)$ is a complementary solution for the differential equation $x^2y'' - 3xy' + 4y = x^3$, find a particular solution satisfying the initial conditions $y(e) = 0, y'(e) = 1$.

The solutions of the homogeneous equation are

$$y_1 = x^2, y_2 = x^2 \ln x$$

The right hand side (after dividing through by x^2) is $f(x) = x$.

$$W(y_1, y_2) = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & 2x \ln x + x \end{vmatrix} = x^3.$$

Using the method of variation of parameters, we have

$$u_1' = \frac{\begin{vmatrix} 0 & x^2 \ln x \\ x & 2x \ln x + x \end{vmatrix}}{x^3} = -\ln x$$

which gives (by using integration by parts)

$$u_1 = -x \ln x + x.$$

Similarly,

$$u_2' = \frac{\begin{vmatrix} x^2 & 0 \\ 2x & x \end{vmatrix}}{x^3} = 1$$

and

$$u_2 = x.$$

Hence,

$$y_p = x^2(-x \ln x + x) + x\left(x^2 \ln x\right) = x^3.$$

The general solution is

$$\begin{aligned} y &= x^2(c_1 + c_2 \ln x) + x^3, \\ y' &= 2c_1x + c_2(2x \ln x + x) + 3x^2 \end{aligned}$$

$y(e) = 0$ *gives*

$$c_1 + c_2 = -e$$

and $y'(e) = 1$ gives

$$2c_1 + 3c_2 = -3e$$

Solving we get

$$c_1 = 0, c_2 = -e.$$

The required solution is

$$y = -ex^2 \ln x + x^3.$$

(b) Find the general solution of the differential equation $y'' + y = 2\cos x$. (*Hint: it may be easy to guess a particular solution.*)

Since the differentiation oparator on the left ($D^2 + 1$) kills both $\sin x$ *and* $\cos x$, *we may try a particular solution of the form*

$$y_p = x\,(A\cos x + B\sin x)$$

Substituting this solution into the equation gives

$$-2A\sin x + 2B\cos x = 2\cos x.$$

Therefore, $A = 0, B = 1$. *The particular solution is* $y_p = x\sin x$ *and the general solution is*

$$y = c_1\cos x + c_2\sin x + x\sin x.$$

3. (a) Show that $\lambda_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$ are the eigenvalues of

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

What are the corresponding eigenvectors?

$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1$. *Using the quadratic formula, we get*

$$\lambda = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

For convenience, let $\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\lambda_2 = \frac{1-\sqrt{5}}{2}$ *and observe that* $\lambda_1 + \lambda_2 = 1$ *and* $\lambda_1\lambda_2 = -1$.

For $\lambda = \lambda_1$, *we have*

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} 1-\lambda_1 & 1 \\ 1 & -\lambda_1 \end{bmatrix} = \begin{bmatrix} \lambda_2 & 1 \\ 1 & -\lambda_1 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & 1/\lambda_2 \\ 1 & -\lambda_1 \end{bmatrix} = \begin{bmatrix} 1 & -\lambda_1 \\ 1 & -\lambda_1 \end{bmatrix}. \end{aligned}$$

Hence

$$v_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$$

Similarly, for $\lambda = \lambda_2$,

$$v_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}.$$

(b) Now consider the model $x_{n+1} = Ax_n$, $n = 0, 1, \ldots$, with $x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and A being the matrix in part (a) of this problem.

1. Show that $x_n = A^n x_0$.

$$\begin{aligned} x_1 &= Ax_0, \\ x_2 &= Ax_1 = AAx_0 = A^2x_0, \\ x_3 &= Ax_2 = AA^2x_0 = A^3x_0 \\ &\vdots \\ x_n &= A^n x_0. \end{aligned}$$

2. Express x_n explicitly in terms of n and compute x_1, x_2.
Write $A = PDP^{-1}$ where

$$P = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, P^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}.$$

Then

$$\begin{aligned} x_n &= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} \begin{bmatrix} 1 - \lambda_2 \\ -1 + \lambda_1 \end{bmatrix} \\ &= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ -\lambda_2 \end{bmatrix} \\ &= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^{n+1} \\ -\lambda_2^{n+1} \end{bmatrix} \\ &= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1^{n+2} - \lambda_2^{n+2} \\ \lambda_1^{n+1} - \lambda_2^{n+1} \end{bmatrix} \end{aligned}$$

3. **(2points)** Show that, for any n, the quantity

$$\frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right]$$

is always an integer.
Since x_0 has integer components and A^n has integral components, it follows that $x_n = A^n x_0$ also has integral components. In particular, the second component of x_n is always an integer. Thus

$$\frac{1}{\lambda_1 - \lambda_2} \left(\lambda_1^{n+1} - \lambda_2^{n+1} \right) = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right]$$

is always an integer.

4. (a) Find the general solution of the linear system

$$x' = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x.$$

(b) **(4points)** Find the solution satisfying the initial condition $x(0) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and compute the solution at $t = 2$.

5. (a) For the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$:

   (b) Show that $A^{2n} = I$ and $A^{2n+1} = A$ for any integer n.

$$A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and

$$\begin{aligned} A^{2n} &= \left(A^2\right)^n = I^n = I, \\ A^{2n+1} &= AA^{2n} = AI = A. \end{aligned}$$

   (c) Use the power series $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ to compute $\cos(At)$.

$$\begin{aligned} \cos(At) &= \sum_{n=0}^{\infty} (-1)^n \frac{(At)^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{A^{2n} t^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{I t^{2n}}{(2n)!} \\ &= \left(\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} \right) I \\ &= \cos t I = \begin{bmatrix} \cos t & 0 \\ 0 & \cos t \end{bmatrix} \end{aligned}$$

   (d) Use the power series $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ to compute $\sin(At)$.
   In a similar way to part c, we compute

$$\begin{aligned} \sin(At) &= \left(\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} \right) A \\ &= \sin t A \\ &= \begin{bmatrix} 0 & \sin t \\ \sin t & 0 \end{bmatrix}. \end{aligned}$$