KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
DEPARTMENT OF MATHEMATICAL SCIENCES
MATH 260
Final Exam, Fall 061
January 29, 2007
Time Allowed: 2 hours

NAME:	ID#:

SHOW ALL YOUR WORK

1. **(10points)** Find a general solution of the differential equations

   (a) $e^x + ye^{xy} + (e^y + xe^{xy})\, y' = 0$

   (b) $xy' + 2y = 6x^2\sqrt{y}$.

2. (a) **(10points)** If $y_c = x^2\left(c_1 + c_2 \ln x\right)$ is a complementary solution for the differential equation $x^2y'' - 3xy' + 4y = x^3$, find a particular solution satisfying the initial conditions $y\left(e\right) = 0, y'\left(e\right) = 1$.

   (b) **(6points)** Find the general solution of the differential equation $y'' + y = 2\cos x$. (*Hint: it may be easy to guess a particular solution.*)

3. (a) **(10points)** Show that $\lambda_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$ are the eigenvalues of

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

What are the corresponding eigenvectors?

(b) Now consider the model $x_{n+1} = Ax_n,\ n = 0, 1, \ldots,$ with $x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and A being the matrix in part (a) of this problem.

1. **(2points)** Show that $x_n = A^n x_0$.
2. **(2points)** Express x_n explicitly in terms of n and compute x_1, x_2.
3. **(2points)** Show that, for any n, the quantity

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

is always an integer.

4. (a) **(10points)** Find the general solution of the linear system

$$x' = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x.$$

(b) **(4points)** Find the solution satisfying the initial condition $x(0) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and compute the solution at $t = 2$.

5. For the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$:

(a) **(8points)** Show that $A^{2n} = I$ and $A^{2n+1} = A$ for any integer n.

(b) **(3points)** Use the power series $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ to compute $\cos(At)$.

(c) **(3points)** Use the power series $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ to compute $\sin(At)$.