KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
DEPARTMENT OF MATHEMATICAL SCIENCES
MATH 260
Exam # 2 Solutions

1. Let A be the matrix

$$A = \begin{bmatrix} 3 & 0 & 11 & -5 & 0 \\ -2 & 4 & 13 & 6 & 5 \\ 0 & 0 & 5 & 0 & 0 \\ 7 & 6 & -9 & 17 & 7 \\ 0 & 0 & 8 & 2 & 0 \end{bmatrix}.$$

   (a) Compute $\det A$

   We can expand using row 3 and get

$$\begin{aligned} \det A &= 5 \begin{vmatrix} 3 & 0 & -5 & 0 \\ -2 & 4 & 6 & 5 \\ 7 & 6 & 17 & 7 \\ 0 & 0 & 2 & 0 \end{vmatrix} \\ &= -10 \begin{vmatrix} 3 & 0 & 0 \\ -2 & 4 & 5 \\ 7 & 6 & 7 \end{vmatrix} \text{ (using the third row above)} \\ &= -30 \begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix} \text{(using the first row above)} \\ &= -30 * -2 = 60. \end{aligned}$$

   (b) If we write $A^{-1} = [a_{ij}]$ find a_{23} and a_{25}.

   Using Cramer's rule,

$$a_{23} = \frac{(-1)^{2+3} M_{32}}{\det A}.$$

   To compute M_{32} it is best to use row operations to simplify the calculation of the determinant. We proceed as follows.

$$
\begin{aligned}
M_{32} &= \begin{vmatrix} 3 & 11 & -5 & 0 \\ -2 & 13 & 6 & 5 \\ 7 & -9 & 17 & 7 \\ 0 & 8 & 2 & 0 \end{vmatrix} \text{(taking common factor from row 4)} \\
&= 2\begin{vmatrix} 3 & 11 & -5 & 0 \\ -2 & 13 & 6 & 5 \\ 7 & -9 & 17 & 7 \\ 0 & 4 & 1 & 0 \end{vmatrix} (C2-4C3) \\
&= 2\begin{vmatrix} 3 & 31 & -5 & 0 \\ -2 & -11 & 6 & 5 \\ 7 & -77 & 17 & 7 \\ 0 & 0 & 1 & 0 \end{vmatrix} \text{(Expanding using } R4) \\
&= -2\begin{vmatrix} 3 & 31 & 0 \\ -2 & -11 & 5 \\ 7 & -77 & 7 \end{vmatrix} (C1-C3, C2+11C3) \\
&= -2\begin{vmatrix} 3 & 31 & 0 \\ -7 & 44 & 5 \\ 0 & 0 & 7 \end{vmatrix} \text{(Expanding using } R3) \\
&= -14\begin{vmatrix} 3 & 31 \\ -7 & 44 \end{vmatrix} = -4886.
\end{aligned}
$$

Therefore, $a_{23} = -\frac{4886}{60} = -81.433$. We compute a_{25} similarly and obtain $a_{25} = \frac{1870}{60} = 31.167$.

2. Use row reduction operations to compute A^{-1} for

$$
A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix}.
$$

We work with the augmented matrix

$$\left[\begin{array}{cccccccc} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R1\leftrightarrows R2}{\longrightarrow} \left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R2\leftrightarrows R3}{\longrightarrow} \left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R3-3R1}{\longrightarrow} \left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 & 0 & 1 \end{array}\right]$$

$$\overset{R2-2R3}{\longrightarrow} \left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 & 0 & 1 \end{array}\right]$$

Therefore,

$$A^{-1} = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 1 \end{array}\right]$$

3. Suppose v_1, v_2, v_3 are linearly independent vectors. Show that the vectors $u_1 = v_2 - v_3, u_2 = v_1 - v_3, u_3 = v_1 - v_2$ are also linearly independent.

Suppose

$$c_1 u_1 + c_2 u_2 + c_3 u_3 = 0.$$

Then

$$\begin{aligned} c_1 (v_2 - v_3) + c_2 (v_1 - v_3) + c_3 (v_1 - v_2) &= 0. \\ (c_2 + c_3) v_1 + (c_1 - c_3) v_2 + (-c_1 - c_2) v_3 &= 0. \end{aligned}$$

Since v_1, v_2, v_3 are linearly independent, we must have

$$\begin{aligned} c_2 + c_3 &= 0 \\ c_1 - c_3 &= 0 \\ -c_1 - c_2 &= 0 \end{aligned}$$

or, in matrix form

$$\left[\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{array}\right] \left[\begin{array}{c} c_1 \\ c_2 \\ c_3 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right].$$

The coefficient matrix has determinant 1. Therefore, the above system has only the trivial solution. Thus $c_1 = c_2 = c_3 = 0$ and the vectors u_1, u_2, u_3 are linearly independent.

4. Show whether or not the vector $w = (2, -3, 2, -3)$ is in the span of the vectors $v_1 = (1, 0, 0, 3)$, $v_2 = (0, 1, -2, 0)$, $v_3 = (0, -1, 1, 1)$.

   If w is in the span of v_1, v_2, v_3, then $w = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$. Finding $\alpha_1, \alpha_2, \alpha_3$ is equivalent to solving the system given by the augmented matrix

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & -2 & 1 & 2 \\ 3 & 0 & 1 & -3 \end{bmatrix}$$

   Using the row reduction operations gives

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & -13 \end{bmatrix}.$$

   This means that the system is inconsistent. Hence, W is not in the span of the vectors v_1, v_2, v_3.

5. Find a basis for the solution space of the homogeneous linear system

$$\begin{aligned} x_1 + 3x_2 - 4x_3 - 8x_4 + 6x_5 &= 0 \\ x_1 \qquad + 2x_3 + x_4 + 3x_5 &= 0 \\ 2x_1 + 7x_2 - 10x_3 - 19x_4 + 13x_5 &= 0 \end{aligned}.$$

   The coefficient matrix is

$$\begin{bmatrix} 1 & 3 & -4 & -8 & 6 \\ 1 & 0 & 2 & 1 & 3 \\ 2 & 7 & -10 & -19 & 13 \end{bmatrix}.$$

   Its echlon form is

$$\begin{bmatrix} 1 & 3 & -4 & -8 & 6 \\ 0 & -3 & 6 & 9 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   Therefore, x_3, x_4, x_5 are free variables. We then get the following basis vectors

$$v_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

6. In parts a and b below determine whether or not the given vectors in $\mathbb{R}^n$ form a basis for $\mathbb{R}^n$. Justify your answers.

(a) $v_1 = (0, 2, -3)\,, v_2 = (7, 4, 11)\,.$

(b) $v_1 = (2, 0, 0, 0)\,, v_2 = (0, 3, 0, 0)\,, v_3 = (0, 0, 7, 6)\,, v_4 = (0, 0, 4, 5)\,.$

Any basis for $\mathbb{R}^n$ must have n vectors which are linearly independent.

(a) We have two vectors in $\mathbb{R}^3$. Therefore, v_1, v_2 do not form a basis for $\mathbb{R}^3$.

(b) We have 4 vectors in $\mathbb{R}^4$. Thus we need only check their linar independence. The matrix

$$[v_1\ v_2\ v_3\ v_4] = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 7 & 6 \\ 0 & 0 & 4 & 5 \end{bmatrix}$$

has determinant $66 \neq 0$. Thus v_1, v_2, v_3, v_4 are linearly independent and they form a basis for $\mathbb{R}^4$.