In problems 1-5 find a general solution for the given differential equation

1. $y' = \sqrt{x+y}$.

   Let $u^2 = x + y$, then $2uu' = 1 + y'$. Substitute; $2uu' - 1 = u$. Separate the variable:

$$\begin{aligned} \frac{2u}{u+1}du &= dx \\ 2\int\left(1 - \frac{1}{u+1}\right)du &= x + C \\ u - \ln(u+1) &= \frac{x}{2} + C \\ \sqrt{x+y} - \ln\left(1 + \sqrt{x+y}\right) &= \frac{x}{2} + C. \end{aligned}$$

2. $xy' + 3y = 3x^{-3/2}$.

$$\begin{aligned} y' + \frac{3}{x}y &= 3x^{-5/2} \\ I.F. &= e^{\int \frac{3}{x}dx} = x^3 \\ \left(x^3 y\right)' &= 3x^{1/2} \\ x^3 y &= 2x^{3/2} + C \\ y &= 2x^{-3/2} + Cx^{-3}. \end{aligned}$$

3. $(e^y + y\cos x) + (xe^y + \sin x)\frac{dy}{dx} = 0$.

   This is an exact equation.

$$\begin{aligned} M &= e^y + y\cos x = F_x. \\ &\therefore\ F = xe^y + y\sin x + h(y) \\ F_y &= xe^y + \sin x + h'(y) = xe^y + \sin x \\ h'(y) &= 0 \\ h(y) &= C \\ F &= xe^y + y\sin x + C. \end{aligned}$$

   The general solution is $xe^y + y\sin x + C = 0$.

4. $y' = \frac{x+3y}{y-3x}$.

This is a homogeneous equation.

$$\begin{aligned}
y' &= \frac{1+3\frac{y}{x}}{\frac{y}{x}-3} \\
u &= \frac{y}{x} \\
xu'+u &= y' \\
xu'+u &= \frac{1+3u}{u-3} \\
xu' &= \frac{1+3u}{u-3}-u \\
&= \frac{6u-u^2+1}{u-3} \\
-\int \frac{u-3}{u^2-6u-1}du &= \int \frac{dx}{x} \\
-\frac{1}{2}\ln\left(u^2-6u-1\right) &= \ln Cx \\
u^2-6u-1 &= \frac{C}{x^2} \\
\left(\frac{y}{x}\right)^2-6\left(\frac{y}{x}\right)-1 &= \frac{C}{x^2} \\
y^2-6xy-x^2 &= C.
\end{aligned}$$

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5. $y'' = (x+y')^2$.

   Let $u = x + y'$.

$$\begin{aligned}
u' &= 1+y'' \\
y'' &= u'-1 \\
u'-1 &= u^2 \\
\int \frac{1}{u^2+1}du &= x+C \\
\tan^{-1}u &= x+C \\
u &= \tan(x+C) \\
y' &= -1+\tan(x+C) \\
y &= -x+\ln|\sec(x+C)|+D.
\end{aligned}$$

6. Solve the differential equation $\frac{dy}{dx} = 3(y+3)x^2$ in two different ways. Do you get the same solutions? Discuss the differences, if any.

First Method: linear equation.

$$\begin{aligned} y' - 3x^2 y &= 9x^2 \\ I.F. &= e^{-\int 3x^2 dx} = e^{-x^3}. \\ \left(e^{-x^3} y\right)' &= 9x^2 e^{-x^3} \\ e^{-x^3} y &= \int 9x^2 e^{-x^3} = -3e^{-x^3} + C \\ y &= -3 + Ce^{x^3}. \end{aligned}$$

Second method: separation of variables.

$$\begin{aligned} \int \frac{dy}{y+3} &= 3\int x^2 dx. \\ \ln|y+3| &= x^3 + C \\ |y+3| &= e^{x^3+C} = e^{x^3} e^C \\ &= Ke^{x^3} \text{ (here } K \text{ is nonnegative)} \\ y+3 &= \pm Ke^{x^3} \\ y &= -3 \pm Ke^{x^3}. \end{aligned}$$

Comment: Following the second solution as written above, we end up with a nonnegative constant K, but $\pm K$ admits positive as well as negative constants. Therefore, we end up with the same general solution as with the first method. If, however, we drop the absolute value bars in the steps of the second solutions, we end up, as before, with

$$y + 3 = Ke^{x^3}$$

where K is nonnegative. The solution in this case is a special case of the solution with the first method where C can be positive as well as negative.