Lecture 6

$\Gamma-$reqularization

Definition: Let $F : V \longrightarrow \bar{\mathbb{R}}$, a function $G \in \Gamma(V)$ is called the Γregularizer of F if G is the pointwise supremum of all caf minorant of F.
Remark:s
* If G is the $\Gamma - reg\ F$, then G is lower semicontinuous and convex.
* note that $G = \Gamma - reg\ F$ iff G is the greatest minorant in $\Gamma(V)$ of F.
* G is a minorant and if $\widetilde{G} \in \Gamma(V)$ is a minorant of F, then $G \geqslant \widetilde{G}$. on the other hand, suppose that G is the greatest minorant of F in
$\Gamma(V)$.Let G_1 be the $\Gamma - reg\ F \Longrightarrow G_1 \geqslant G$, but by hypothesis $G \geqslant G_1 \Longrightarrow G = G_1$.

Proposition: Let $F : V \longrightarrow \bar{\mathbb{R}}, F$ has a caf minorant, $G = \Gamma - reg\ F \Longrightarrow epiG = \overline{co}\ epiF$
Proof:
$epiG \supseteq \overline{co}\ epiF$. on the other hand, assume $\left(\overline{\overline{v}},\ \overline{a}\ \right) \notin \overline{co}\ epiF$ $\Longrightarrow$there exists a caf $l(u) + \alpha a + \beta$ strictly separating $\left(\overline{\overline{v}},\ \overline{a}\ \right)$ and $\overline{co}\ epiF$.

$\therefore l(\overline{\overline{v}})+\alpha\overline{a}+\beta < 0 and l(v)+\alpha a+\beta > 0 \forall (v,a) \in \overline{co} epiF. \left(\overline{\overline{v}},\ F(\ \overline{\overline{v}})\right) \in epiF \subseteq \overline{co} epiF \Longrightarrow l(\overline{\overline{v}})+\alpha F(\ \overline{\overline{v}})+\beta > 0 and -l(\overline{\overline{v}})-\alpha\overline{a}-\beta > 0$

adding these inequalities

$$\Longrightarrow \alpha(F(\ \overline{\overline{v}}) - \overline{a}) > 0 \Longrightarrow \alpha > 0$$

for

$(v, F(v) \in epiF \Longrightarrow l(v)+\alpha F(v)+\beta > 0 \Longrightarrow F(v) > \dfrac{-1}{\alpha}(l(v)+\beta) \Longrightarrow G(v) > \dfrac{-1}{\alpha}(l(v)+\beta)\ \forall\ v \in domF \Longrightarrow G(\overline{v}) > \dfrac{-1}{\alpha}(l(\overline{v})+\beta) >$

remark:
$F : V \longrightarrow \bar{\mathbb{R}}, \overline{F} = lsc\ reqF$ and $G = \Gamma - reg\ F \Longrightarrow$
1) $G \leq \overline{F} \leq F$
2) if F is convex, with one caf minorant, then $G = \overline{F}$. indeed;

$$F \text{ is convex} \Longrightarrow \overline{F} \text{ convex , } epi\ \overline{F} = \overline{epiF},\ \overline{F} \in \Gamma(V) \Longrightarrow \overline{F} \leq G \Longrightarrow G = \overline{F}$$

1.4 polar Functions

Let $F : V \longrightarrow \bar{\mathbb{R}}$,suppose $\left\langle u, \overset{*}{u} \right\rangle - \alpha$ is a minorant of F. the polar function $\overset{*}{F} : \overset{*}{V} \longrightarrow \bar{\mathbb{R}}$ is defined by

$$\overset{*}{F}(\overset{*}{u}) = \sup_{u \in V} \left\langle u, \overset{*}{u} \right\rangle - F(u)$$