Lecture 4

Continuity of Convex Function

PROPOSITION 1
$F : V \longrightarrow \bar{R}$,If F is Convex and bounded above in a nbhd of a point $u \in V$, then F is continuous at u.

Proof. Assume $u = 0$, and $F(0) = 0$, let W be nbhd of 0 and F is bounded by $a < \infty$ on W. Let $W_1 = W \cap -W$, let $\epsilon > 0$ be given, let $v \in \epsilon W_1$.

$$F(v) = F(\frac{\epsilon v}{\epsilon}) \leq \epsilon F(\frac{v}{\epsilon}) \leq \epsilon a,$$

also,$-v \in W_1$

$$0 = \frac{1}{2}v - \frac{1}{2}v$$

$$0 = F(0) \leq \frac{1}{2}F(v) + \frac{1}{2}F(-v) \Longrightarrow$$

$$-F(v) \leq F(-v) = F(\epsilon\frac{-v}{\epsilon}) \leq \epsilon F(\frac{-v}{\epsilon}) \leq \epsilon a,$$

then

$$|F(v)| \leq \epsilon a \quad \Longrightarrow F \text{ is continuous at } 0.$$

■

PROPOSITION 2
Let $F : V \rightarrow \bar{R}$ be a convex function, TFAE

(i) $\exists$ an open, non-empty $O \subseteq V$, s.t. F is bounded above (by $a < \infty$) on V and $F(O) \neq \{-\infty\}$.

(ii) $\overset{o}{\widehat{domF}} \neq \phi$, F is continuous and proper on $\overset{o}{\widehat{domF}}$.

Proof. Clearly (ii)$\Longrightarrow$(i). Conversely for (i)$\Longrightarrow$(ii), $\overset{o}{\widehat{domF}} \neq \phi$ since $O \subsetneq domF$.Let $u \in \overset{o}{\widehat{domF}}$ and choose $v \in O$ s.t. $|F(v)| < \infty$. since u is a internal point of the convex set $\overset{o}{\widehat{domF}}$, there exists a $w_1 \in \overset{o}{\widehat{domF}}$ s.t. $u \in (w_1, v)$

$$u = \alpha w_1 + (1 - \alpha)v \in \alpha w_1 + (1 - \alpha)O,$$

let $z \in \alpha w_1 + (1 - \alpha)O$

$$z = \alpha w_1 + (1 - \alpha)z_2 \text{ where } z_2 \in O,$$

$$F(z) \leq \alpha F(w_1) + (1 - \alpha)F(z_2) \leq \alpha F(w_1) + (1 - \alpha)a$$

therefore F is bounded above on the open nbhd $\alpha w_1 + (1 - \alpha)O$ of u. Then F is continuous at u. ■

COROLLARY 3
$F : V \rightarrow R$ convex, V is finite dimension, then F is continuous on $\overset{o}{\widehat{domF}}$.

Proof. If $\overset{o}{\widehat{domF}} \neq \phi$, then $\overset{o}{\widehat{domF}}$ contains an interior point. $\overset{o}{\widehat{domF}}$ contains $(n + 1)$ affinely independent vectors $(u_1, u_2, .., u_{n+1})$. For $u \in \overset{o}{\widehat{domF}}$, there exists an open set of the form $I_1 \times I_2 \times ...I_n$, u can be written as $u = \sum_{i=1}^{n+1} \lambda_i u_i$ s.t. $0 \leq \lambda_i \leq 1$ and $\sum_{i=1}^{n+1} \lambda_i = 1$, then

$$F(u) \leq \sum_{i=1}^{n+1} \lambda_i F(u_i) \leq \sum_{i=1}^{n+1} F(u_i),$$

therefor F is bounded above on a nbhd of u. ■

COROLLARY 4
Let V be a normed space, $F : V \rightarrow \bar{R}$ is a a proper convex function. TFAE:

(i) $\exists$ an open set $O \subseteq V$ on which F is bounded in O.

(ii) $\overset{o}{\widehat{domF}} \neq \phi$, and F is locally Lipschitz on $\overset{o}{\widehat{domF}}$.