Lecture 3

Recall that a function $F : V \longrightarrow \overline{I\!R}$ is lower semicontinuous if

$$\liminf_{u \to \bar{u}} F(u) \geq F(\bar{u})$$

Lemma 1
F is lsc iff $S_a = \{u \in V : F(u) \leq a\}$ is closed in V.

Proof. The necessary condition was done in the previous lecture. For sufficient condition, suppose that S_a for all $a \in I\!R$ and let $\bar{u} \in V$ and $a = \liminf_{u \to \bar{u}} F(u)$.
Case 1: If $a = \infty$ then nothing to prove.
Case 2: If a is finite ($\|a\| < \infty$), take a sequence $\{u_n\}$ such that $u_n \to \bar{u}$. For each k, we can find n_k such that

$$F(u_{n_k}) \leq a + \frac{1}{k}$$

these $u_{n_k} \in S_{a+\frac{1}{k}}$ and we have

$$u_{n_k} \in \bigcap_{i=1}^{k} S_{a+\frac{1}{i}}$$

and since $\bigcap_{i=1}^{k} S_{a+\frac{1}{i}}$ is closed and $u_{n_k} \to \bar{u}$ then

$$u \in \bigcap_{i=1}^{\infty} S_{a+\frac{1}{i}} = S_a$$

$\therefore F(\bar{u}) \leq a = \liminf_{u \to \bar{u}} F(u)$
Case 3: If $a = -\infty$ consider $S_n = \{u \in V : F(u) \leq -n\}$. ■

Proposition 2
F is lsc iff $epiF$ is closed.

Proof. Let $\phi : V \times I\!R \longrightarrow \overline{I\!R}$ be defined by

$$\phi(u, a) = F(u) - a$$

Now, let $(u_n, a_n) \to (u, a)$; that is $u_n \to u$ and $a_n \to a$. Then

$$\liminf \phi(u_n, a_n) = \liminf F(u_n) - a_n \geq F(u) - a = \phi(u, a)$$

So $\phi(u, a)$ is lsc, then by previous lemma the set $\{(u, a) : \phi(u, a) \leq \alpha\}$ is closed for each $\alpha \in \overline{I\!R}$. In particular, if $\alpha = 0$, the set

$$\{(u, a) : \phi(u, a) \leq 0\} = \{(u, a) : F(u) \leq a\}$$

is closed; which is the epigraph of f.
Now suppose that $epiF$ is closed. Then the set

$$\{(u, a) \in V \times I\!R : \phi(u, a) \leq r\} = \{(u, a) \in V \times I\!R : F(u) \leq a + r\} = \{(u, a) \in V \times I\!R : F(u) \leq a\} - \{(0, r) : r \in I\!R\}$$

is closed. Therefore, ϕ is lsc. It remains to show that F is lsc if ϕ is. For this let $\{u_n\}$ be a sequence such that $u_n \to \bar{u}$ and consider

$$\liminf_{u_n \to \bar{u}} F(u_n) - \quad a = \liminf F(u_n) - \liminf a = \liminf \left(F(u_n) - a\right) = \liminf \phi(u_n, a) \geq \phi(u, a) = F(u) - a$$

Therefore F is lsc. ■

Lemma 3
If $(F_i)_{i \in I}$ is a family of lsc functions, then $F(u) = \sup_{i \in I} F_i(u)$ is as well lsc.

Proof. Claim: $epiF = \bigcap epiF_i$. To show this,

$$\begin{aligned} \text{let } (u,a) \in epiF &\Leftrightarrow F(u) \le a \\ &\Leftrightarrow \sup F_i(u) \le a \\ &\Leftrightarrow F_i(u) \le a \qquad \forall i \\ &\Leftrightarrow (u,a) \in epiF_i \qquad \forall i \\ &\Leftrightarrow (u,a) \in \bigcap epiF_i \end{aligned}$$

So F is lsc. ∎

Definition 4
A function $\bar{F}$ is called the lsc regularization of F if it is the greatest lsc minorant of F (i.e. $\bar{F}(u) \le F(u)$ for all $u \in V$).

Theorem 5 If $F : V \longrightarrow \overline{I\!R}$. Then

(a) $epi\bar{F} = \overline{epiF}$.

(b) $\bar{F}(u) = \liminf F(u)$.

Proof.

(a) Read the book.

(b) Let $(u,a) \in epi\bar{F}$, then $(u,a) \in \overline{epiF}$ and there exists a sequence (u_n, a_n) in $epiF$ such that $(u_n, a_n) \leftarrow (u,a)$. Now for each n we have $\bar{F}(u_n) \le F(u_n) \le a_n$ and

$$\bar{F}(u) \le \liminf \bar{F}(u_n) \le \liminf F(u_n) \le \liminf a_n = a = \bar{F}(u)$$

Therefore $\bar{F}(u) = \liminf F(u)$ as desired. ∎

Corollary 6
The function $F : V \longrightarrow \overline{I\!R}$ is lsc and convex iff F is weakly lsc and convex.

Proof.
F is lsc and convex$\Leftrightarrow epiF$ is convex and closed
$\Leftrightarrow epiF$ is the intersection of all half spaces containing it.
$\Leftrightarrow epiF$ is weakly lsc and convex.
$\Leftrightarrow F$ is weakly lsc and convex.
which concludes the proof. ∎

Proposition 7
If $F : V \longrightarrow \overline{I\!R}$ is lsc and convex and $F(\bar{u}) = -\infty$ for some $\bar{u} \in V$, then F can not take any finite value.

Proof. Assume $|F(u)| < \infty$. Let $u_n = \alpha_n \bar{u} + (1-\alpha_n)u, \alpha_n \leftarrow 0$ then

$$F(u_n) = F(\alpha_n \bar{u} + (1-\alpha_n)u) \le \alpha_n F(\bar{u}) + (1-\alpha_n)F(u) = -\infty$$

which is a contradiction. ∎