Lecture 21

Applications of Duality to the calculus of varitions

Preliminaries

Let $\Omega \subseteq \mathbb{R}^n$ be open, sometimes we require reqularity on Ω.
Regularity: Ω is said to be of class C^r if the boundary Γ is an $r-$times continuously differential maiifold of dimention $(n-1)$ and Ω lies locally in one side of Γ.
For $x \in \Gamma$, $\nu(x) = (\nu_1(x), \nu_2(x), ..., \nu_n(x))$ will denote the outward normal to Ω.

Differentiation, Multiindex Notation.
for $j = (j_1, j_2, ..., j_n) \in \mathbb{N}^n$,
$D^j u = D^j D^{j_2} ... D^{j_n} u = \frac{\partial^{|j|}}{\partial x_1^{j_1} \partial x_2^{j_2} ... \partial x_n^{j_n}}$ where $|j| = j_1 + j_2 + ... + j_n$.
Examle: let $j = (1, 2, 4, 0) \in \mathbb{N}^4$,
$D^j u = \frac{\partial^7 u}{\partial x_1 \partial x_2^2 \partial x_3^4}$

Remark: $D^{(0,0,...,0)} = I$
Space $L^\alpha(\Omega), 1 \leq \alpha < \infty$
$L^\alpha(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} : \int\limits_\Omega |u(x)|^\alpha \, dx < \infty \right\}$is a Banach space under the norm $\|u\|_{L^\alpha(\Omega)} = \left(\int\limits_\Omega |u(x)|^\alpha \, dx \right)^{\frac{1}{\alpha}}$.
Space $L^\infty(\Omega)$
$L^\infty(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} : Ess. \sup\limits_{x \in \Omega} |u(x)| < \infty \right\}$ is a Banach space under the norm $\|u\|_{L^\infty(\Omega)} = Ess. \sup\limits_{x \in \Omega} |u(x)|$

The Dual spaces of $L^\alpha(\Omega)$
$(L^\alpha(\Omega))^* = L^{\alpha'}(\Omega)$ where $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$
Special case: if $\alpha = 2 \Longrightarrow \alpha' = 2, L^2(\Omega)$ is a Hilbert space with inner product $\langle u, v \rangle = \int\limits_\Omega u(x) v(x) dx$
The Soblev Spaces $\overset{m,\alpha}{w}(\Omega), \overset{m,\alpha}{w_0}(\Omega)$ where $1 \leq \alpha < \infty$ and $m \geqslant 1$ is an integer.
$\overset{m,\alpha}{w}(\Omega) = \{ u \in L^\alpha(\Omega) : D^k u \in L^\alpha(\Omega), |k| \leq m \}$ is a Banach Space under the norm $\|u\|_{\overset{m,\alpha}{w}(\Omega)} = \left(\sum\limits_{|j| \leq m} \int\limits_\Omega |D^j u(x)|^\alpha \, dx \right)^{\frac{1}{\alpha}}$
$\overset{m,\alpha}{w_0}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm of $\overset{m,\alpha}{w}(\Omega)$.
The Trace Operator : suppose $\Omega \in C^{m+2}$
The operator $\gamma : (\gamma_0, \gamma_1, ..., \gamma_{m-1}) : \overset{m,\alpha}{w}(\Omega) \longrightarrow L^\alpha(\Gamma)$ defined by
$\gamma_0 u = u|_\Gamma, \gamma_1 u = \frac{\partial u}{\partial \nu}|_\Gamma .... \gamma_{n-1} u = \frac{\partial^{m-1} u}{\partial \nu^{m-1}}|_\Gamma$where $\frac{\partial u}{\partial \nu} = \nabla u . \nu|_\Gamma$and $\frac{\partial^k u}{\partial \nu^k} = \frac{\partial}{\partial \nu} \frac{\partial^{k-1} u}{\partial \nu^{k-1}}|_\Gamma = \nabla \left(\frac{\partial^{k-1} u}{\partial \nu^{k-1}} \right) . \nu|_\Gamma$is called the Trace Operator.
γ is linear and continuous operator, also $Ker \ \gamma = \overset{m,\alpha}{w_0}(\Omega)$

Poincare' Inequality (assume Ω to be bounded)

For all $u \in \overset{1,\alpha}{w_0}(\Omega)$, $\|u\|_{L^\alpha(\Omega)} \leq c \|D^i u\|_{L^\alpha(\Omega)}$ where c is a constant depends on Ω and α. i.e $c(\Omega, \alpha)$.
Green's Formula (Integration by Parts)
let $u \in \overset{1,\alpha}{w}(\Omega)$ and $v \in \overset{1,\alpha'}{w}(\Omega)$, then $\int\limits_\Gamma u v \nu_i d \, \Gamma = \int\limits_\Omega (u D_i v + v D_i u) \, dx$ (1) where ν_i is the i_{th} component of ν.
if we replace v by $D_i v$ in (1)
$\int\limits_\Gamma u \ D_i v \ \nu_i d\Gamma = \int\limits_\Omega (u D_i^2 v + D_i v D_i u) \, dx$, sum for $i = 1, 2, ..., n$, we get $\int\limits_\Gamma u \ \frac{\partial v}{\partial \nu} d \, \Gamma = \int\limits_\Omega (u \bigtriangleup v + \nabla u . \nabla v) \, dx$
also if interchanged u and v we get $\int\limits_\Gamma v \ \frac{\partial u}{\partial \nu} d \, \Gamma = \int\limits_\Omega (v \bigtriangleup u + \nabla u . \nabla v) \, dx$ subtracting we get, $\int\limits_\Gamma (u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu}) d \, \Gamma = \int\limits_\Omega (u \bigtriangleup v - v \bigtriangleup u) \, dx$
also, replace v by v_i in (1)v $\Longrightarrow \int\limits_\Gamma u \ v_i \ \nu_i d \, \Gamma = \int\limits_\Omega (u D_i v_i + v_i D_i u) \, dx$ or $\int\limits_\Gamma u \ v . \ \nu d \, \Gamma = \int\limits_\Omega (u \nabla . v + v . \nabla u) \, dx$.