Lecture 20 (Important Special Case (II))

The dual problem

For $p^* \in Y$,

$$\Phi^*(0,p^*) = \sup_{u\in V} \sup_{p\in Y} < p,p^* > -\Phi(u,p)$$

$$= \sup_{u\in V} \sup_{p\in Y} < p,p^* > -\hat{J}(u) - \chi_\epsilon(u,p)$$

$$= \sup_{u\in A} \sup_{Bu\leq p} < p,p^* > -J(u);$$

Let $q = p - Bu$,we get

$$\Phi^*(0,p^*) = \sup \ \sup < q + Bu, p^* > -J(u)$$

$$= \sup_{u\in A} \sup_{q\geq 0} < q,p^* > + < Bu,p^* > -J(u)$$

$$= \sup_{u\in A} \ < Bu,p^* > -J(u) + \sup_{q\geq 0} < q,p^* >$$

$$= \sup_{u\in A} \ < Bu,p^* > -J(u) + \chi_{C^*}(-p),$$

then,

$$-\Phi^*(0,p^*) = \inf_{u\in A} \ - < Bu,p^* > +J(u) - \chi_{C^*}(-p),$$

Thus the dual problem is

$$P^* \qquad \sup_{p^*\in Y^*} \inf_{u\in A} \ - < Bu,p^* > +J(u) - \chi_{C^*}(-p)$$
$$\sup_{p^*\leq 0} \inf_{u\in A} \ - < Bu,p^* > +J(u).$$

Stability

$\inf P \in \mathbf{R}$,for some $u_0 \in A$, $Bu \in -C^\circ$(the interior of C).Then P is stable.

Existence

Assume V is a reflexive Banach space, $J(u) \to \infty$ as $\|u\| \to \infty, u \in A$, Then P has a solution.

Extremality

$$\inf P = \sup P^*,$$

the extremality relation

$$< B\bar{u}, \bar{p}^* >= 0.$$

because:

$$\inf P = J(\bar{u}), \qquad \bar{u} \in A, \ B\bar{u} \leq 0,$$

$$\sup P^* = \inf_{u\in A} - < Bu, \bar{p}^* > J(u) \ , \qquad \bar{p}^* < 0 \qquad (*)$$

and

$$J(\bar{u}) = \inf_{u\in A} - < Bu, \bar{p}^* > +J(u) \leq - < B\bar{u}, \bar{p}^* > +J(\bar{u}) \qquad (**)$$

then we have

$$< B\bar{u}, \bar{p}^* > \leq 0,$$

from (*) and (**) we have

$$< B\bar{u}, \bar{p}^* > \geq 0,$$

Then,we have the extremality relation

$$< B\bar{u}, \bar{p}^* >= 0.$$

The Lagrangian

$$-L(u,p^*) = \sup_{p\in Y} < p, p^* > -\Phi(u,p)$$

$$= \sup_{p\in Y} < p, p^* > -\hat{J}(u) - \chi_\epsilon(u,p)$$

$$= -\hat{J}(u) + \sup_{Bu\leq p} < p, p^* >$$

$$= -\hat{J}(u) + \sup_{q\geq 0} < Bu, p^* > - < q, p^* >$$

$$= -\hat{J}(u) + < Bu, p^* > + \chi_{C^*}(-p^*).$$

Then,

$$L(u,p^*) = \hat{J}(u) - < Bu, p^* > - \chi_{C^*}(-p^*).$$

Proposition $(\bar{u},\bar{p}^*) \in V\times Y^*$ is a saddle point of L if and only if $\bar{u}\in A, \bar{p}^* < 0$,and

$$J(\bar{u}) - < B\bar{u}, p^* > \leq J(\bar{u}) - < B\bar{u}, \bar{p}^* > \leq J(u) - < Bu, \bar{p}^* >, \quad \forall u\in A, \forall p^*\leq 0. \tag{(1)}$$

Proof: assume $(\bar{u},\bar{p}^*)$ is a saddle point of L, (let $u\in A$ and $p^*\leq 0$)

$$- < B\bar{u}, p^* > + \hat{J}(\bar{u}) - \chi_{C^*}(-p^*) \leq - < B\bar{u}, \bar{p}^* > + \hat{J}(\bar{u}) - \chi_{C^*}(-\bar{p}^*)$$

$$\leq - < Bu, \bar{p}^* > + \hat{J}(u) - \chi_{C^*}(-\bar{p}^*),$$

then

$$-\infty < - < B\bar{u}, p^* > + \hat{J}(\bar{u}) \leq - < B\bar{u}, \bar{p}^* > + \hat{J}(\bar{u}) - \chi_{C^*}(-\bar{p}^*)$$
$$\leq - < Bu, \bar{p}^* > + \hat{J}(u) - \chi_{C^*}(-\bar{p}^*),$$

the left most and right most parts of the inequalities give $\bar{p}^*\leq 0$,and the second and the third parts give $\bar{u}\in A$.

$$- < B\bar{u}, p^* > + \hat{J}(\bar{u}) \leq - < B\bar{u}, \bar{p}^* > + \hat{J}(\bar{u}) \leq - < Bu, \bar{p}^* > + \hat{J}(u).$$

Assume $\bar{u}\in A$ and $\bar{p}^*\leq 0$ and (1) is satisfied,

$$L(\bar{u},\bar{p}^*) = - < B\bar{u}, \bar{p}^* > + \hat{J}(\bar{u}),$$

$$L(u,\bar{p}^*) = - < Bu, \bar{p}^* > + \hat{J}(u),$$

$$L(\bar{u},p^*) = - < B\bar{u}, p^* > + \hat{J}(\bar{u}) - \chi_{C^*}(-p^*),$$

then

$$L(\bar{u},p^*) \leq L(\bar{u},\bar{p}^*) \leq L(u,\bar{p}^*),$$

then $(\bar{u},\bar{p}^*)$ is a saddle point of L.

Kuhn-Tucker theorem $V = V^* = \mathbf{R}^n, Y = Y^* = \mathbf{R}^m, A\subseteq \mathbf{R}^n$ is closed convex set.

$$J : A\to \mathbf{R}, \quad \text{convex and } l.s.c.$$

the cone C,

$$C = \{p\in \mathbf{R}^m : p_i\geq 0, i = 1,2,...,m\}.$$

$C^* = C$,
the function $B : A\to \mathbf{R}^m$ is defined by $Bu = (B_1u, B_2u, ..., B_mu)$, and

$$B_i : A\to \mathbf{R} \quad \text{convex and } l.s.c.$$

$$B_iu_0 < 0, \quad i = 1,2,.....m \text{ for some } u_0\in A.$$

the primal problem is

$$P \qquad \inf_{u\in A, Bu\leq 0} J(u)$$

$\bar{u} \in A$ is a solution of P iff there exists $\bar{p} \in \mathbf{R}^m, \bar{p} \leq 0$ such that $(\bar{u}, \bar{p})$ is a saddle point of $L,$ in this case

$$\sum_{i=1}^{m} p_i B_i \bar{u} = 0,$$

note that P is stable, if $\bar{u}$ is a solution of P therefor P^* has a solution $\bar{p} \leq 0$,and $(\bar{u}, \bar{p})$ is a saddle point of L.On the other hand if $\bar{p} \leq 0$ such that $(\bar{u}, \bar{p})$ is a saddle point of L, $\bar{u}$ is a solution of P. By the previous proposition, $\bar{u} \in A$.

$$\bar{p} \leq 0 \Rightarrow \bar{p}_i \leq 0 \quad \forall i$$

$$B\bar{u} \leq 0 \Rightarrow B_i\bar{u} \leq 0 \quad \forall i$$

$$\sum_{i=1}^{m} p_i B_i \bar{u} = 0 \Rightarrow p_i B_i \bar{u} = 0,$$

if $B_i\bar{u} < 0$ then $p_i = 0$ and if $p_i < 0$ then $B_i\bar{u} = 0.$