Lecture 19 (Important Special Cases II)

Definition 1 *Let C be a subset of a linear space Y, then C is*

- *a cone if $\lambda C \subset C$ for all $\lambda > 0$.*
- *a pointed cone if it is a cone containing zero.*
- *a salient cone if it is a pointed cone with $C \cap (-C) = \{0\}$.*

Definition 2 *A cone C of a linear space Y induces a partial ordering defined by $p \geq 0$ iff $p \in C$.*

This means if $p \leq q$, then $q - p \in C$. If C is salient, then $\leq$ is an ordering relation. If $\leq$ is an ordering relation on Y compatible with the linear structure of Y (That is: $\lambda p \leq \lambda q, \forall \lambda > 0$ and $p + v \leq q + v, \forall v \in Y$ if $p \leq q$). Then $\{p \in Y : p \geq 0\}$ is a salient pointed cone.

Definition 3 *The polar cone of a cone C is the set*

$$C^* = \{p^* \in Y^* : \langle p^*, p\rangle \geq 0 \forall p \in C\}$$

Lemma 4 *If C is a convex pointed cone, then*

(i) C^* is closed (in $\sigma(Y^*, Y)$).

(ii) $C^{**} = C$.

(iii) $p \in C$ iff $p \in C$ iff $\langle p^*, p\rangle \geq 0$ for all $p^* \in C^*$.

Proof.

1. To show that C^* is closed, we write

$$\begin{aligned} C^* &= \{p^* \in Y^* : \langle p^*, p\rangle \geq 0 \forall p \in C\} \\ &= \bigcap_{p \in C} \{p^* \in Y^* : \langle p^*, p\rangle \geq 0\} \\ &= \bigcap_{p \in C} \{p^{-1}[0, \infty)\} \end{aligned}$$

   Since p is continuous in the topology $\sigma(Y^*, Y)$; C is closed.

2. $C \subset C^{**}$ is clear. To show that $C^{**} \subset C$; let $q \in C^{**}$, then $\langle q, p^*\rangle \geq 0$ for all $p^* \in C^*$. Assume that $q \notin C$, so there exists $x \neq 0 \in Y^*$ such that $\langle x, p\rangle \geq \alpha$ for all $p \in C$ and $\alpha \in I\!R$ and $\langle x, q\rangle < \alpha$. Since $0 \in C$, then $\alpha \leq 0$. Hence $\langle x, q\rangle < 0$, but this can not happen; since $x \in C^*$. To show that, assume otherwise then there exists $p' \in C$ such that $\langle x, p'\rangle < 0 \Rightarrow \lambda\langle x, p'\rangle = \langle x, \lambda p'\rangle < 0$. But for sufficiently small λ, we have $\langle x, \lambda p'\rangle < \alpha$ which is a contradiction. So $x \in C^*$, but again this is a contradiction. Thus $q \in C$.

3. $p \in C \Rightarrow p \geq 0 \Rightarrow \langle p^*, p\rangle \geq 0 \forall p^* \in C^* \Rightarrow p \in C^{**} = C$.

■

The problem considered

Let $\phi \neq A \subset V$ be closed and convex, $J : V \longrightarrow I\!R$ convex and lsc, C closed convex cone in Y, $\leq$ the partial ordering induced by C. $B : A \longrightarrow Y$ satisfy the following:

(B1) B is convex with respect to $\leq$.

(B2) For each $p^* \in C^*$, $\langle p^*, B(\cdot)\rangle : A \longrightarrow I\!R$ is lsc.

(B3) The set $\{u \in A : B(u) \leq 0\} \neq \phi$.

Primal problem

$$\inf_{\substack{u \in A \\ Bu \le 0}} \quad J(u)$$

Perturbation problem

$$\Phi(u,p) = \begin{cases} J(u) & \text{if } u \in A, Bu \le p, \\ +\infty & \text{otherwise .} \end{cases}$$

Lemma 5 *The set $\mathcal{E} = \{(u,p) \in V \times Y : u \in A, Bu \le p\}$ is closed and convex.*

Proof.

$$\begin{aligned} \mathcal{E} &= \{(u,p) \in V \times Y : u \in A, \langle p^*, Bu - p\rangle \le 0 \forall p^* \in C\} \\ &= \bigcap_{p^* \in C^*} \{(u,p) \in V \times Y : u \in A, \langle p^*, Bu - p\rangle \le 0\} \cap (A \times Y) \end{aligned}$$

which is closed; since $u \longmapsto \langle p^*, Bu - p\rangle$ is lsc by (B2). To show the convexity of C, let $(u,p),(v,q) \in \mathcal{E}$ where $u, v \in A$ and $p, q \in Y$ and $\lambda \in [0,1]$. Then

$$\lambda(u,p) + (1-\lambda)(v,q) = (\lambda u + (1-\lambda)v, \lambda p + (1-\lambda)q)$$

Since A is convex $\lambda u + (1-\lambda)v \in A$. Now B is convex

$$B[\lambda u + (1-\lambda)v] \le \lambda Bu + (1-\lambda)Bv \le \lambda p + (1-\lambda)q.$$

Hence $\lambda u + (1-\lambda)v \in \mathcal{E}$ which proves that $\mathcal{E}$ is convex. ■
We can rewrite ϕ as

$$\phi(u,p) = \hat{J}(u) = \chi_{\mathcal{E}} \quad \text{where } \hat{J}(u) = \begin{cases} J(u), & u \in A \\ +\infty, u \notin A \end{cases}$$

Proposition 6 $\phi \in \Gamma_0(V \times Y)$

1. *ϕ does not take the value $-\infty$.*
2. *$\phi \not\equiv +\infty (\phi(u,0) < +\infty)$.*
3. *ϕ is convex.*
4. *ϕ is lsc.*