Lecture 17

Summery
P: $\underset{u\in v}{inf} F(u)$
$\Phi : V \times Y \longrightarrow \overline{R}$ such that $\Phi(u,0) = F(u)$
P: $\underset{u\in v}{inf} \Phi(u,0)$
The dual problem

$\overset{*}{P}$: $\underset{\overset{*}{p}\in\overset{*}{y}}{sup} - \overset{*}{\Phi}(0,\overset{*}{p})$
$Sup\ \overset{*}{P} \le \inf P$

$-\overset{*}{\Phi}(0,\overset{*}{p}) \le \sup \overset{*}{P} \le \inf P \le \Phi(u,0) \Rightarrow \Phi(u,0) + \overset{*}{\Phi}(0,\overset{*}{p}) \ge 0$
$h(p) = \inf_{u\in v} \Phi(u,p)$
- If $h(0) \in \mathbb{R}$ and h is lwoer semicontinuous at $0 \Rightarrow P$ is normal
- P is normal $\Leftrightarrow \inf P = \sup \overset{*}{p} \Leftrightarrow \overset{*}{p}$ is normal
- $h(0) \in \mathbb{R}$, $\partial h(0) \neq \Phi \Longrightarrow$ P is stable
- P is stable iff $\overset{*}{P}$ is normal and has some solutions
- the set of solution of $\overset{*}{P}$ conicides with $\partial \overset{**}{h}(0)$
- $P, \overset{*}{P}$ are normal and have same solutions $\Leftrightarrow P$ and $\overset{*}{P}$ are stable $\Leftrightarrow$ P is stable and has solutions.

Criterion for stability
Φ is convex, $h(0) \in \mathbb{R}$, $\Phi(u,.)$ bounded above in a nbhd of 0 $\Longrightarrow P$ is stable
$h(p) \le \Phi(u,p)$

Criterion for existence
V is a reflexive Banach space, $\Phi(.,0)$ is coercive $\Longrightarrow P$ has a solution

Extremality relation and Existence

Lemma1: $\overline{u} \in V$ is a solution of P and $\overset{*}{\overline{p}}$ is a solution of $\overset{*}{P}$ and $\inf P = \sup \overset{*}{P}$ iff $\Phi(\overline{u},0) + \overset{*}{\Phi}(0,\overset{*}{\overline{p}}) = 0$
Proof: if $\overline{u} \in V$ is a solution of P and $\overset{*}{\overline{p}}$ is a solution of $\overset{*}{P}$ and $\inf P = \sup \overset{*}{P}$, then $-\overset{*}{\Phi}(0,\overset{*}{\overline{p}}) = \sup \overset{*}{P}$ $= \inf P = \Phi(\overline{u},0) \Longrightarrow \Phi(\overline{u},0) + \overset{*}{\Phi}(0,\overset{*}{\overline{p}}) = 0$
conversly assume $\Phi(\overline{u},0) + \overset{*}{\Phi}(0,\overset{*}{\overline{p}}) = 0$ for some $\overline{u} \in V$ and some $\overset{*}{\overline{p}} \in \overset{*}{Y}$ then

$$-\overset{*}{\Phi}(0,\overset{*}{\overline{p}}) \le \sup \overset{*}{P} \le \inf P \le \Phi(\overline{u},0) = -\overset{*}{\Phi}(0,\overset{*}{\overline{p}})$$

and hence, the result is obtained.

Lagrangians and Saddle points

Definition: $L : V \times \overset{*}{Y} \longrightarrow \overline{\mathbb{R}}$ defined by $-L(u,\overset{*}{P}) = \underset{p\in Y}{Sup} \left\langle p, \overset{*}{p} \right\rangle - \Phi(u,p)$ is called the Lagrangian .

Note: $-L(u,\overset{*}{P}) = \overset{*}{\Phi}_u(\overset{*}{p})$ where $\Phi_u(p) = \Phi(u,p)$
Lemma
1- for $u \in V$,$L(u,.)$ is concave and u.s.c.

2- if Φ is convex, then for any $\overset{*}{p} \in \overset{*}{Y}, L(.,\overset{*}{P})$ is convex

Proof: (part 2)

$L(\lambda u+(1-\lambda)v,\overset{*}{p}) = \inf_{p\in Y} -\left\langle p,\overset{*}{p}\right\rangle + \Phi((\lambda u+(1-\lambda)v,p) \leq -\left\langle \lambda p+(1-\lambda)q,\overset{*}{p}\right\rangle + \Phi((\lambda u+(1-\lambda)v,\lambda p+(1-\lambda)q) \leq$
$\lambda(-\left\langle p,\overset{*}{p}\right\rangle + \Phi(u,p)) + (1-\lambda)(-\left\langle q,\overset{*}{p}\right\rangle + \Phi(u,q))$
fix q and take the inf over $p \Longrightarrow$
$L(\lambda u+(1-\lambda)v,\overset{*}{p}) \leq \lambda L(u,\overset{*}{p}) + (1-\lambda)(-\left\langle q,\overset{*}{p}\right\rangle + \Phi(u,q))$
now take inf over $q \Longrightarrow$
$L(\lambda u+(1-\lambda)v,\overset{*}{p}) \leq \lambda L(u,\overset{*}{p}) + (1-\lambda)L(v,\overset{*}{p})$ and hence, $L(.,\overset{*}{P})$ is convex.

$\overset{*}{P}$ in terms of L
$\overset{*}{\Phi}(\overset{*}{u},\overset{*}{p}) = \sup_{u\in v, p\in Y} \left\langle u,\overset{*}{u}\right\rangle + \left\langle p,\overset{*}{p}\right\rangle - \Phi(u,p)$

$\overset{*}{\Phi}(0,\overset{*}{p}) = \sup_{u\in V}\sup_{p\in Y} \left\langle p,\overset{*}{p}\right\rangle - \Phi(u,p) = \sup_{u\in V} - L(u,\overset{*}{p}) = -\inf_{u\in V} L(u,\overset{*}{p}) \Longrightarrow -\overset{*}{\Phi}(0,\overset{*}{p}) = \inf_{u\in V} L(u,\overset{*}{p}) \Longrightarrow \overset{*}{P}:$
$\sup_{\overset{*}{p}\in\overset{*}{Y}} \inf_{u\in V} L(u,\overset{*}{p})$
P in terms of L
$\Phi(u,0) = \Phi_u(0) = \sup_{\overset{*}{p}\in\overset{*}{Y}} \left\langle 0,\overset{*}{p}\right\rangle - \overset{*}{\Phi}_u(\overset{*}{p}) = \sup_{\overset{*}{p}\in\overset{*}{Y}} - \overset{*}{\Phi}_u(\overset{*}{p}) = \sup_{\overset{*}{p}\in\overset{*}{Y}} L(u,\overset{*}{p}) \Longrightarrow P: \inf_{u\in V}\sup_{\overset{*}{p}\in\overset{*}{Y}} L(u,\overset{*}{p})$
Definition: (Saddle point)
$(\overline{u},\overset{*}{\overline{p}}) \in V\times\overset{*}{Y}$ is called a saddle point of L if $L(\overline{u},\overset{*}{p}) \leq L(\overline{u},\overset{*}{\overline{p}}) \leq L(u,\overset{*}{\overline{p}})$ for all $u\in V, \overset{*}{p}\in\overset{*}{Y}$.

Lemma2:$(\overline{u},\overset{*}{\overline{p}}) \in V\times\overset{*}{Y}$ is called a saddle point of L iff $\overline{u}$ is a solution of P and $\overset{*}{\overline{p}}$ is a solution of $\overset{*}{P}$ and $\inf P = \sup \overset{*}{P}$
Proof:
($\Longrightarrow$)assume $(\overline{u},\overset{*}{\overline{p}})$ is a saddle point $\Longrightarrow \Phi(\overline{u},0) = \sup_{\overset{*}{p}\in\overset{*}{Y}} L(\overline{u},\overset{*}{p}) \leq L(\overline{u},\overset{*}{\overline{p}}) \leq \inf_{u\in V} L(u,\overset{*}{\overline{p}}) = -\overset{*}{\Phi}(0,\overset{*}{\overline{p}}) \Longrightarrow \Phi(\overline{u},0) +$
$\overset{*}{\Phi}(0,\overset{*}{\overline{p}}) \leq 0$ but $\Phi(\overline{u},0) + \overset{*}{\Phi}(0,\overset{*}{\overline{p}}) \geqslant 0$
$\Longrightarrow \Phi(\overline{u},0) + \overset{*}{\Phi}(0,\overset{*}{\overline{p}}) = 0$ and we get the extremality condition, so $\inf P = \sup \overset{*}{P}$.
($\Longleftarrow$) assume $\overline{u}$ is a solution of P and $\overset{*}{\overline{p}}$ is a solution of $\overset{*}{P}$ and $\inf P = \sup \overset{*}{P}$
$\Phi(\overline{u},0) = \sup_{\overset{*}{p}\in\overset{*}{Y}} L(\overline{u},\overset{*}{p}) \geqslant L(\overline{u},\overset{*}{\overline{p}}) \geqslant \inf_{u\in V} L(u,\overset{*}{\overline{p}}) = -\overset{*}{\Phi}(0,\overset{*}{\overline{p}})$
$L(u,\overset{*}{\overline{p}}) \geqslant \inf_{u\in V} L(u,\overset{*}{\overline{p}}) = L(\overline{u},\overset{*}{\overline{p}}) = \sup_{\overset{*}{p}\in\overset{*}{Y}} L(\overline{u},\overset{*}{p}) \geqslant L(\overline{u},\overset{*}{p})$ and hence, $(\overline{u},\overset{*}{\overline{p}})$ is a saddle point.