Lecture 16

(Stable Problems)

Definition 1 *Problem P is called stable if* $h(0) \in \mathbf{R}$,$\partial h(0) \neq \phi$.

Lemma 2 *The set of solution of* $\mathbf{P}^*$ *coincides with* $\partial h^{**}(0)$.

Proof. Suppose $p^*$is a solution of $\mathbf{P}^*$, then

$$-h^*(p^*) = -\Phi(0,p^*) = \sup_{q^* \in Y} - \Phi(0,q^*) = h^{**}(0).$$

Fix $p \in Y$, then,

$$\sup_{q^* \in Y} < p, q^* > -h^*(q^*) \geq h^*(p^*) + < p^*, p >$$

i.e.:

$$h^{**}(p) \geq -h^*(p^*) + < p^*, p > = h^{**}(0) + < p^*, p >$$

Then, $p^* \in \partial h^{**}(0)$.
On the other hand, let $p^* \in \partial h^{**}(0)$,then

$$h^{**}(p) \geq h^{**}(0) + < p^*, p > \quad \forall v \in V$$

$$-h^{**}(0) \geq < p^*, p > -h^{**}(p)$$
$$-h^{**}(0) \geq h^{***}(p^*) = h^*(p^*)$$
$$h^{**}(0) \leq -h^*(p^*)$$
$$\sup -h^*(q^*) \leq -h^*(p^*)$$

Therefore,

$$-h^*(p^*) = \sup -h^*(q^*) \qquad q^* \in Y$$

Then, $p^*$is a solution of $\mathbf{P}^*$. ■

Proposition 3 $\mathbf{P}$ *is stable iff* $\mathbf{P}$ *is normal and* $\mathbf{P}^*$ *has a solution.*

Proof. Suppose $\mathbf{P}$ is stable, then $\mathbf{P}$ is normal (since $\partial h(0) \neq 0 \implies h$ is $l.s.c$ at 0). Furthermore, $p^* \in \partial h(0) = \partial h^{**}(0)$, therefore, $p^*$is a solution of $\mathbf{P}^*$ by previous lemma. Conversely if $\mathbf{P}$ is normal and $\mathbf{P}^*$ has a solution p^*, then

$$p^* \in \partial h^{**}(0) = \partial h(0),$$

since h is $l.s.c$ at 0.Then $\mathbf{P}$ is stable. ■

Proposition 4 *The Following Conditions are equivalent:*
(I) $\mathbf{P}$ *and* $\mathbf{P}^*$*are normal and have some solutions,*
(II) $\mathbf{P}$ *and* $\mathbf{P}^*$*are stable,*
(III) $\mathbf{P}$ *is stable and has some solutions.*

Proof. (I) $\Rightarrow$(II),
Assume (I) , $\mathbf{P}^*$ is normal and $\mathbf{P}$ has a solution $\Rightarrow$ $\mathbf{P}^*$ is normal and $\mathbf{P}^{**}$ has a solution,$\Rightarrow$ $\mathbf{P}^*$is stable. Similarly, $\mathbf{P}$ is normal and $\mathbf{P}^*$ has a solution $\Rightarrow$ $\mathbf{P}$ is stable. (II) $\Rightarrow$(I) direct. (III) $\Rightarrow$(I) follows directly from previous proposition. ■

Proposition 5 *A stability criterion.*
Assume Φ *is convex, that* $\inf \mathbf{P} \in \mathbf{R}$. $\Phi(u_0, .)$ *is bounded above at* 0 *for some* $u_0 \in \mathbf{V}$. *Then* $\mathbf{P}$ *is stable.*

Proof.

$$h(p) = \inf_{u \in \mathbf{V}} \Phi(u,p) \leq \Phi(u_0, p),$$

and $h(0) \in \mathbf{R} \Rightarrow h$ is bounded above at $0, \Rightarrow h$ is continuous at $0, \Rightarrow \partial h(0) \neq \phi$. Then $\mathbf{P}$ is stable. ■