Lecture 15 (Duality in convex optimization)

Setting: V, Y are topological vector spaces, V^*, Y^* are their dual, $F : V \longrightarrow \bar{R}$ and

$$(P) \qquad \inf_{u \in V} F(u)$$

- The $\inf$ for problem (P) will be denoted by $\inf P$.
- A solution of (P) is any $u \in V$ such that $F(u) = \inf P$.
- Problem (P) is called nontrivial if $\exists u_0 \in V$ such that $F(u_0) < \infty$. If $F \in \Gamma_0(V)$, then (P) is nontrivial.

Suppose $\Phi : V \times Y \longrightarrow \bar{R}$ such that $\Phi(u, 0) = F(u)$. The problem

$$(P_p) \qquad \inf_{u \in V} \Phi(u, p)$$

is called the perturbed prolem of (P) with respect to Φ ($P_0 = P$). The problem

$$(P^*) \qquad \sup_{p^* \in Y^*} \{-\Phi(0, p^*)\}$$

is called the dual of (P) with respect to Φ [1].

Proposition 1

$$-\infty \leq \sup P^* \leq \inf P \leq \infty$$

Proof. $\sup P^* = \sup_{p^* \in V^*} \{-\Phi^*(0, p^*)\}$

$$\begin{aligned} \Phi^*(0, p^*) &= \sup_{(u,p) \in V \times Y)} \{\langle p, p^* \rangle - \Phi(u, p)\} \\ &\geq \sup_{u \in V} -\Phi(u, 0) \\ &= -\inf_{u \in V} F(u) \end{aligned}$$

So, $\sup P^* \leq \inf P$. ■

Proposition 2 *If P is nontrivial then*

$$-\infty \leq \sup P^* \leq \inf P < \infty$$

If P^* is nontrivial then

$$-\infty < \sup P^* \leq \inf P \leq \infty$$

If P and P^* are nontrivial then

$$-\infty < \sup P^* \leq \inf P < \infty$$

Reiteration of duality

The problem

$$(P^*_{u^*}) \qquad \sup_{p^* \in Y^*} \{-\Phi(u^*, p^*)\}$$

is called the associated perturbed problem of P^*. The bidual problem

$$(P^{**}) \qquad \inf_{u \in V} \{\Phi^{**}(u, 0)\}$$

This process terminates. Indeed, $P^{***} = P^*$.

- If $P^{**} = P$ ($Q^{**} = Q$), then P, P^* are the dual of each other.
- If $\Phi \in \Gamma(V, Y)$ then $P^{**} = P$ and P is nontrivial.

[1] $\Phi : V \times Y \longrightarrow \bar{R}$, $\langle (v^*, p^*), (v, p) \rangle = \langle v, v^* \rangle + \langle p, p^* \rangle$

$$\Phi^*(v^*, p^*) = \sup_{(v,p) \in V \times Y} \langle (v^*, p^*), (v, p) \rangle - \Phi(u, p) = \sup_{(v,p) \in V \times Y} \langle v, v^* \rangle + \langle p, p^* \rangle - \Phi(u, p)$$

Normal problems and stable problems

$\Phi \in \Gamma_0(V \times Y)$ define $h(p) = \inf P_p = \inf \Phi(u,p)$.

Lemma 3 *$h : Y \longrightarrow I\!R$ is convex.*

Proof. Let $p, q \in Y$ and $\lambda \in [0,1]$. Assume that $\lambda h(p) + (1-\lambda)h(q)$ is defined. If either $h(p)$ or $h(q)$ is infinite, nothing to prove. Assume $h(p)$ and $h(q)$ are finite. Let $\epsilon > 0$ be given, there exists a $u_1 \in V$ such that

$$\Phi(u,p) \leq h(p) + \epsilon$$

and there exists $u_2 \in V$ such that

$$\Phi(u,q) \leq h(q) + \epsilon$$

Now, we have

$$\begin{aligned} h\left[\lambda h(p) + (1-\lambda)h(q)\right] &\leq Q\left[\lambda(u_1,p) + (1-\lambda)(u_2,q)\right] \\ &\leq \lambda Q(u_1,p) + (1-\lambda)Q(u_2,q) \\ &\leq \lambda h(p) + (1-\lambda)h(q) + \epsilon \end{aligned}$$

Since ϵ is arbitrary h is convex. ∎

Lemma 4 *For all $p^* \in V^*$*

$$h^*(p^*) = \Phi^*(0,p^*)$$

Proof.

$$\begin{aligned} h^*(p^*) &= \sup_{p \in Y} \langle p^*, p\rangle - h(p) \\ &= \sup_{p \in Y} \{\langle p^*, p\rangle - \inf_{u \in V} \Phi(u,p)\} \\ &= \sup_{(u,p) \in V \times Y} \{\langle p^*, p\rangle - \Phi(u,p)\} \\ &= \sup_{(u,p) \in V \times Y} \{\langle u, 0\rangle + \langle p^*, p\rangle - \Phi(u,p)\} = \Phi^*(0,p^*) \end{aligned}$$

∎

Lemma 5 $\sup P^* = h^{**}(0)$.

Proof.

$$\begin{aligned} \sup P^* &= \sup_{p^* \in Y^*} \{-\Phi^*(0,p^*)\} \\ &= \sup_{p^* \in Y^*} \{-h^*(p^*)\} \\ &= \sup_{p^* \in Y^*} \{\langle 0, p^*\rangle - h^*(p^*)\} = \Phi(0,p^*) \end{aligned}$$

∎

Remark 6

$$\sup P^* \leq \inf P \Leftrightarrow h^{**}(0) \leq h(0)$$

Definition 7 *The problem (P) is called normal if $h(0) \in I\!R$ and h is lsc at 0.*

Proposition 8 *Problem (P) is normal iff $\sup P^* = inf P \in I\!R$.*

Proof. Assume that (P) is normal. Let $\overline{h}$ be the lsc regularization of h. Then

$$h^{**} \leq \overline{h} \leq h \quad (1)$$

$\overline{h}(0) = h(0)$, $\overline{h}$ is convex, lsc and finite at 0. So

$$\overline{h} \not\equiv -\infty \Rightarrow \overline{h} \in \Gamma_0(Y) \Rightarrow \overline{h}^{**} = \overline{h}$$

From 1

$$h^{*} \leq \overline{h}^{*} \leq h^{***} = h^{*}$$

but $h^{*} = \overline{h}^{*}$. So $h^{**} = \overline{h}^{**} = \overline{h}$ and $h^{**}(0) = \overline{h}(0) = h(0)$. That is

$$\sup P^{*} = inf P$$

.

Now assume $\sup P^{*} = inf P \in I\!R$. Then $h^{**}(0) = h(0)$. Let $\overline{h}$ be the lsc regularization of h

$$h^{**} \leq \overline{h} \leq h$$

So h is lsc at 0, i.e.

$$h(0) = \overline{h}(0) = \liminf_{p \longrightarrow 0} h(p)$$

■

Lemma 9 P^{*} *is normal iff* $\inf P = \sup P^{*}$

Proof. By proposition (8) P^{*} is normal iff $\inf P^{**} = \sup P^{*}$ i.e. $\inf P = \sup P^{*}$ ■