1 Minimization of Convex Functions and Variational Inequalities

Recall that :

1. a normed vector space X is called reflexive if $X = X^{**}$.
2. A Banach space is reflexive if its unit ball is compact in the weak topology.
3. Hilbert spaces and L^p spaces $(1 < p < \infty)$ are reflexive.

Let V be a reflexive Banach space (with norm$\|\|$) and $\phi \neq C$ is closed convex subset of V. The function $F : C \to \mathbf{R}$, is convex and $l.s.c$ and proper. $\hat{F} : V \to \bar{\mathbf{R}}$ is the convex extension of F to all V.

$$\hat{F}(u) = \begin{cases} F(u) & \text{if } u \in C \\ +\infty & \text{if } u \notin C \end{cases}$$

$\hat{F}$ is convex and $l.s.c.$
Consider the minimization problem:

$$\alpha = \inf_{v \in C} F(v) = \inf_{v \in V} \hat{F}(v) \qquad ((*))$$

Definition 1 *an element $u \in C$, s.t. $F(u) = \alpha$ is called a solution of the problem (*).*

Proposition 2 (1) *The set of solution of (*) is closed and convex set (possibly empty).*

Proof. Proof. *Consider the set*

$$\left\{ u \in V : \hat{F}(u) \leq \alpha \right\}$$

since $\hat{F}$ is convex and $l.s.c$ the set is convex and closed. ■ ■

Proposition 3 (2) *If C is bounded or F is coercive , then (*) has at least one solution. It has a unique solution if F is strictly convex.*

Proof. Let $\{u_n\}$ be a sequence in C s.t.

$$F(u_n) \to \alpha = \inf_{v \in C} F(v)$$

- If C is bounded then $\{u_n\}$ is bounded.
- If F is coercive then $F(u_n) \to \alpha \neq \infty$, then$F(u_n)$ is bounded above, the subsequence $\{u_{n_k}\} \overset{weakly}{\to} u.$
- C is closed $\Rightarrow$ C is weakly closed $\Rightarrow$ $u \in C$.
- F is convex and $l.s.c \Rightarrow F$ weakly $l.s.c.$
- $F(u_n) \leq \underline{\lim} F(u_n) = \lim F(u_n) = \alpha >$
- Then $F(u) = \alpha$. u is a solution.

■

Proposition 4 (3) *Consider $F : C \to \mathbf{R}$,$F' exists, u \in C$, The following are Equivalent:*

(i)

$$u \text{ minimizes } F \ \ on \ \ C.$$

(ii)

$$< F'(u), v - u > \geq 0 \quad \forall v \in C.$$

(iii)

$$< F'(v), v - u > \geq 0 \quad \forall v \in C.$$

Proof. (i)$\Rightarrow$ (ii)
$< F'(u), v - u > = \lim_{\gimel \to 0} \frac{F(u+\gimel(v-u))-F(u)}{\gimel} \geq 0$
(ii)$\Rightarrow$ (iii)

$$F(u) \geq F(v) + < F'(v), u - v >$$
$$F(v) \geq F(u) + < F'(u), v - u >$$

Adding them

$$0 \geq < F'(v), u - v > + < F'(u), v - u >$$
$$< F'(v), v - u > \geq < F'(u), v - u > \geq 0$$

(iii)$\Rightarrow$ (ii)

$$F(v) \geq F(\gimel u + (1-\gimel)v) + < F'(\gimel u + (1-\gimel)v), \gimel(v-u) >$$
$$= F(\gimel u + (1-\gimel)v) + \frac{\gimel}{1-\gimel} < F'(\gimel u + (1-\gimel)v), (1-\gimel)(v-u) >$$
$$\geq F(\gimel u + (1-\gimel)v) = \phi(\gimel)$$
$$F(v) \geq \phi(1) = F(u)$$
$$\Rightarrow F(u) \text{ is a minimum.}$$

■

Remark 5 $F(u) = a(u,u) - 2 < l, v >$

- $a(.,.)$ is continuous bilinear form ($|a(u,v)| \leq \|u\| \, \|v\|$) ,
- $a(u,u) \geq \gamma \|u\|^2, \gamma > 0.$
- $l \in V^*$ (continuous linear functional)
- F is strictly convex , Coercive, Then F has a unique minimum.
- if $u \neq v$

$$a(u,v) + a(v,u) < a(u,u) + a(v,v)$$
$$0 < a(u-v, u-v)$$

- F is strictly convex, $u \neq v$, $\lambda \in (0,1)$,

$$F(\gimel u + (1-\gimel)v) = a(\gimel u + (1-\gimel)v, \gimel u + (1-\gimel)v - 2 < l, \gimel u + (1-\gimel)v >$$
$$= \gimel^2 a(u,u) + \gimel(1-\gimel)(a(u,v) + a(v,u)) + (1-\gimel)^2 a(v,v)$$
$$-2\gimel < l, u > -2(1-\gimel) < l, v >$$
$$< \gimel^2 a(u,u) + \gimel(1-\gimel)(a(u,u) + a(v,u) + (1-\gimel)^2 a(v,v)$$
$$-2\gimel < l, u > -2(1-\gimel) < l, v >$$
$$= \gimel a(u,u) + \gimel(1-\gimel) a(v,v) - 2\gimel < l, u > -2(1-\gimel) < l, v >$$
$$\Rightarrow F \text{ is strictly convex}$$

- F is coercive,

$$\begin{aligned} F(u) &= a(u,u) - 2 < l, u > \\ &\geq \gamma \|u\|^2 - 2 < l, u > \\ &\geq \gamma \|u\|^2 - 2 \|l\| \|u\| \to \infty, \text{ as } \|u\| \to \infty \end{aligned}$$

Then we have a unique minima.

- If F is considered on a bounded set C, then we only required $a(u,u) > 0$.