Lecture 10

Let $F : A \subseteq V \longrightarrow I\!R$, where A is convex. F' exists on A. F is convex iff

$$F(v) \geq F(u) + \langle F'(u), v-u \rangle, \qquad \forall \quad u, v \in A$$

proposition proof continued. Let $u, v \in A$

$$F(v) \geq F[u+\lambda(v-u)] + (1-\lambda)\langle F'[u+\lambda(v-u)], v-u\rangle \qquad (1)$$
$$F(u) \geq F[u+\lambda(v-u)] + \lambda\langle F'[u+\lambda(v-u)], u-v\rangle \qquad (2)$$

Multiplying (1) by λ and (2) by $1-\lambda$ and adding we get

$$F[(1-\lambda)u + \lambda v] \leq (1-\lambda)F(u) + \lambda F(v)$$

which completes the proof of the proposition. ■

PROPOSITION 1
Let $F : A \subseteq V \longrightarrow I\!R$, A is convex, F' exists on A. Then F is convex F' is monotone. That is

$$\langle F'(u) - F'(v), u-v\rangle \geq 0, \qquad \forall \quad u, v \in A$$

Subdifferential Calculus

Let $F : V \longrightarrow \bar{I\!R}$. Then

- $\partial(\lambda F)(u) = \lambda \partial F(u), \qquad \forall \quad \lambda > 0.$
- $\partial(F_1 + F_2)(u) \supseteq \partial F_1(u) + \partial F_2(u).$

Now choose $u^* \in \partial F_1(u), v^* \in \partial F_2(u)$

$$\begin{array}{lcll} F_1(v) & \geq & F_1(u) + \langle v-u, u^*\rangle, & \forall \quad v \in V \\ F_2(v) & \geq & F_2(u) + \langle v-u, v^*\rangle, & \forall \quad v \in V \\ \hline \text{Adding} & & & \\ (F_1+F_2)(v) & \geq & (F_1+F_2)(u) + \langle v-u, u^*+v^*\rangle, & \forall \quad v \in V \end{array}$$

PROPOSITION 2
Let $F_1, F_2 \in \Gamma(V), \bar{u} \in \text{dom}F_1 \cap \text{dom}F_2, F_1$ is continuous at $\bar{u}$, then

$$\partial(F_1+F_2)(u) = \partial F_1(u) + \partial F_2(u)$$

Proof. Let $u^* \in \partial(F_1+F_2)(u)$. Then

$$-\langle v-u, u^*+v^*\rangle - F_1(u) + F_1(v) \geq F_2(u) - F_2(v)$$

Let $G(v) = -\langle v-u, u^*+v^*\rangle - F_1(u) + F_1(v)$ and define

$$\begin{array}{lcl} C_1 & = & \{(v,a) \in V \times I\!R : G(v) \leq a\} = \text{epi}G \\ C_1 & = & \{(v,a) \in V \times I\!R : F_2(u) - F_2(v) \geq a\} \end{array}$$

$\overset{\circ}{C_1} \neq \phi, \overset{\circ}{C_1} \cap C_2 = \phi$ (If not, let $(v,a) \in \overset{\circ}{C_1} \cap C_2$. Then $G(v) < a$ and $F_2(u) - F_2(v) \geq a$). Therefore there exist $v^* \in V^*, \alpha, \beta \in I\!R$ such that

$$\begin{array}{lcll} \langle v, v^*\rangle + \alpha a + \beta & \geq & 0, & \forall \quad (v,a) \in C_1 \\ \langle v, v^*\rangle + \alpha a + \beta & \leq & 0, & \forall \quad (v,a) \in C_2 \end{array}$$

Since $(u,0) \in C_1 \cap C_2$. Then $\langle u, v^*\rangle + \beta = 0 \Rightarrow \beta = -\langle u, v^*\rangle$ and

$$\begin{array}{lcll} \langle v-u, v^*\rangle + \alpha a & \geq & 0, & \forall \quad (v,a) \in C_1 \\ \langle v, v^*\rangle + \alpha a & \leq & 0, & \forall \quad (v,a) \in C_2 \end{array}$$

We can show that $\alpha > 0$. Now for $(v, G(v)) \in C_1$, we have

$$\langle v-u, v^* \rangle + \alpha G(v) \geq 0 \Rightarrow G(v) \geq \langle v-u, -\frac{1}{\alpha}v^* \rangle$$

That is

$$-\langle v-u, u^*+v^* \rangle - F_1(u) + F_1(v) \geq \langle v-u, -\frac{1}{\alpha}v^* \rangle \Rightarrow F_1(v) \geq F_1(u) + \langle v-u, u^* - \frac{1}{\alpha}v^* \rangle$$

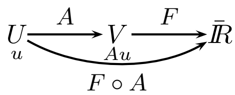
Thus $u^* - \frac{1}{\alpha}v^* \in \partial F_1(u)$. On the other hand, for $(v, F_2(u) - F_2(v)) \in C_2$

$$\langle v-u, v^* \rangle + \alpha(F_2(u) - F_2(v)) \leq 0 \Rightarrow F_2(v) \geq F_2(u) + \langle v-u, \frac{1}{\alpha}v^* \rangle$$

Therefore $\frac{1}{\alpha}v^* \in \partial F_2(u)$ and so $u^* \in \partial F_1(u) + \partial F_2(u)$. ■

PROPOSITION 3
$A : U \longrightarrow V$ is a continuous linear operator, $F \in \Gamma(V)$. If F is continuous and finite at Au, then

$$\partial F \circ A = A^* \partial F(Au)$$



Proof. Suppose $u^* \in A^*\partial F(Au)$ and let $u^* = A^*v^*$ where $v^* \in \partial F(Au)$. Then

$$F(v) \geq F(Au) + \langle v - Au, v^* \rangle, \qquad \forall \quad v \in V$$

In particular for $v = Aw, w \in U$

$$F(Aw) \geq F(Au) + \langle Aw - Au, v^* \rangle, \qquad \forall \quad w \in U$$

So,

$$(F \circ A)(w) \geq (F \circ A) + \langle w - u, A^*v^* \rangle$$

Therefore $A^*v^* = u^* \in \partial\partial(F \circ A)(u)$.
Conversely, let $u^* \in \partial(F \circ A)(u)$. Then

$$(F \circ A)(v) \geq (F \circ A)(u) + \langle v - u, u^* \rangle, \qquad \forall \quad v \in U$$

Let $C_1 = \{(Av, \langle v-u, u^* \rangle + F(Au)) : v \in U\}$. Clearly C_1 is convex and $C_1 \cap \overset{\circ}{\overbrace{\text{epi}F}} = \phi$. Hence there exist $v^* \in V^*, \alpha, \beta \in I\!R$ such that

$$\begin{array}{rcll} \langle v, v^* \rangle + \alpha a + \beta & \geq & 0 & \forall \quad (v,a) \in \text{epi}F \\ \langle v, v^* \rangle + \alpha a + \beta & \leq & 0 & \forall \quad (v,a) \in C_1 \end{array}$$

Now for $(Au, F(Au)))$ we get

$$\begin{array}{rcl} \langle Au, v^* \rangle + \alpha F(Au) + \beta & = & 0 \\ \beta & = & -\langle Au, v^* \rangle - \alpha F(Au) \end{array}$$

So

$$\begin{array}{rcll} \langle v - Au, v^* \rangle + \alpha(a - F(Au))) & \geq & 0 & \forall \quad (v,a) \in \text{epi}F \\ \langle v - Au, v^* \rangle + \alpha(a - F(Au))) & \leq & 0 & \forall \quad (v,a) \in C_1 \end{array}$$

We can show in the same manner as before that $\alpha > 0$. Since $(Av, \langle v-u, u^* \rangle + F(Au)) \in C_1$ we have

$$\langle Av - Au, v^* \rangle + \alpha\langle v-u, u^* \rangle \leq 0 \Rightarrow \langle v-u, A^*v^* - \alpha u^* \rangle \leq 0 \quad \forall \quad \in V.$$

Therefore $A^*v^* + \alpha u^* = 0$ (since a linear functional that keeps the same sign for the whole space must be zero). So

$$u^* = A^*(-\frac{1}{\alpha}v^*) \in A^*\partial F(Au)$$

Which completes the proof. ■