Lecture 5

<u>Theorem</u>:
Let V be a real vector space and let $F : V \to \overline{R}$.Then the following are equivalent:

(1) $\exists\, \emptyset \neq O \subseteq V$ such that F is bounded above in O.
(2)$\overset{\circ}{\widehat{dom}}\ F \neq \emptyset$, F is locally Lipschitz on $\overset{\circ}{\widehat{dom}}\ F$

<u>Proof</u>:
$(1) \Rightarrow (2)$
Let $u \in \overset{\circ}{\widehat{dom}}\ F$. Then, F is continuous at u. So F is absolutely bouned (by a) in a ball $\overline{B(u,r)}$, $r > 0$. Let $v \in B(u,r)$. Write $v = (1-\lambda)u + \lambda w_1$

$$\begin{aligned}
\Rightarrow \quad & v - u = \lambda(w_1 - u) \\
\Rightarrow \quad & \|v - u\| = \lambda r \\
\Rightarrow \quad & F(v) - F(u) = F((1-\lambda)u + \lambda w_1) - F(u) \\
& \leq (1-\lambda)F(u) + \lambda F(w_1) - F(u) \\
& = \lambda(F(w_1) - F(u)) < 2a\tfrac{\|u-v\|}{r}
\end{aligned}$$

Now if $u = (1-\overline{\lambda})v + \overline{\lambda} w_2$

$$\begin{aligned}
\Rightarrow \quad & u - v = \overline{\lambda}(w_2 - v) \quad \Rightarrow \overline{\lambda} = \tfrac{\|u-v\|}{r+\|u-v\|} \\
& F(u) - F(v) \leq 2a\overline{\lambda} = 2a\tfrac{\|u-v\|}{r+\|u-v\|} \leq \tfrac{2a}{r}\|u-v\| \quad \Rightarrow \\
& |\ F(u) - F(v)\ | \leq \tfrac{2a}{r}\|u-v\|
\end{aligned}$$

For any $v \in \overset{\circ}{\widehat{dom}}\ F$ cover $[u,v]$ by a finite set $B(u_i, r_i)$, $i = 1, 2, ..., n$ for which
$u_1 = u, \quad u_n = v$ and $u_{i+1} \in B(u_i, r_i)$. Then,

$$\begin{aligned}
|\ F(u) - F(v)\ | &\leq \sum_{i=1}^{n-1} |\ F(u_{i+1}) - F(u_i)\ | \\
&\leq \sum_{i=1}^{n-1} \tfrac{2a_i}{r_i}\|u_{i+1} - u_i\| \\
&\leq \sum_{i=1}^{n-1} \tfrac{2a_i}{r_i} c_i \|u - v\|
\end{aligned}$$

where $c_i = \frac{\|u_{i+1} - u_i\|}{\|u-v\|}$

Definition:(Cafs)
A caf is the pointwise (pw) supermum of a continuous affine fanuctionals.

Definition:($\Gamma(V)$)
$\Gamma(V)$ is the set of funtions $F : V \to \overline{R}$ which are the pw superma of families of cafs.

Note:
(1) ∞ and $-\infty \in \Gamma(V)$
(2) $\Gamma_\circ(V) = \Gamma(V) \backslash \{-\infty, \infty\}$.
(3) $F \in \Gamma(V) \Rightarrow F$ is convex and l.s.c.

<u>Proposition:</u>
The following are equivalent:

(i) $F \in \Gamma(V)$
(ii) F is convex and l.s.c. and if F assumes the value of $-\infty$, then $F \equiv -\infty$

Proof:
(ii)⇒(i)
Suppose that F is convex and l.s.c. If $F \equiv -\infty,\ F \in \Gamma(V)$ and if $F \equiv \infty,\ F \in \Gamma(V)$.
If F is proper and (F is not $\equiv \infty$). Let $u \in V$. Then we have two cases:

Case(1): $F(u) < \infty$
Let $\overline{a} < F(u)$. Then $\exists$ a hyperplane $H : L(v) + \alpha a + \beta = 0 \quad \forall\, v \in V$ that strictly separate $epi\ F$ and $(u, \overline{a})$. i.e.

$$L(v) + \alpha a + \beta > 0 \quad \forall \ (v,a) \in epi\ F \quad \text{and}$$
$$L(v) + \alpha \overline{a} + \beta < 0$$

Claim that $\alpha > 0$.
For $(u, F(u)) \in epi\ F$ we have $L(u) + \alpha F(u) + \beta > 0$
and $-L(u) - \alpha\overline{a} - \beta > 0$
$\Rightarrow \ \alpha(F(u) - \overline{a}) > 0 \quad \Rightarrow \ \alpha > 0$

So, $F(v) > -\frac{1}{\alpha}(L(v) + \beta) \quad \forall\, v \in V$
$\Rightarrow \qquad \overline{a} < -\frac{1}{\alpha}(L(u) + \beta) < F(u)$

Case (2): $F(u) = \infty$
This means that $\exists$ a hyperplane $H : L(u) + \alpha a + \beta = 0$ that strictly separate $epi\ F$ and $(u, \overline{a})$.If $\alpha \neq 0$, we are back to case (1).
If $\alpha = 0$, then $H : L(u) + \beta = 0$
and $L(u) + \beta < 0$ if we substitute with $(u, \overline{a})$.
From case(1) we can find a caf minorant $m(v) + \gamma$

$$F(v) \geq m(v) + \gamma \quad \forall\, v \in V$$
$$\therefore \ F(v) \geq m(v) + \gamma - c(L(v) + \beta) \quad \forall\, c \geq 0$$

We want to choose c such that

$$m(u) + \gamma - c(L(u) + \beta) > \overline{a}$$
$$c > \frac{\overline{a} - m(u) - \gamma}{-(L(u) + \beta)}$$
$$\Rightarrow \ F \in \Gamma(V)$$

end of Lec# 5