Lecture 29

Existence of Saddle points

Proposition 1
Assume V, Y are reflexive Banach spaces. $A \subset V$, $B \subset Y$ are convex, closed and nonempty. $L : A \times B \longrightarrow \mathbb{R}$.
(1) $L(u, .)$ is concave and upper semicontinuous for each $u \in A$
(2) $L(., p)$ is convex and lower semicontinuous for each $p \in B$.
(3) If A and B are bounded, then L possesses at least one saddle point $(\overline{u}, \overline{p}) \in A \times B$ such that

$$L(\overline{u}, \overline{p}) = \underset{p \in B}{Max} \; \underset{u \in A}{Min} \; L(u, p) = \underset{u \in A}{Min} \; \underset{p \in B}{Max} L(u, p)$$

Proposition 2
If insted of (3) we have
(4)
a) there exists a $p_0 \in B$ such that $L(u, p_0) \longrightarrow \infty$ as $\|u\| \longrightarrow \infty$, $u \in A$
b) there exists a$u_0 \in A$ such that $L(u_0, p) \longrightarrow -\infty$ as $\|p\| \longrightarrow \infty$, $p \in B$,
then L possesses at least one saddle point $(\overline{u}, \overline{p}) \in A \times B$ such that

$$L(\overline{u}, \overline{p}) = \underset{u \in A}{Min} \; \underset{p \in B}{Sup} \; L(u, p) = \underset{p \in B}{Max} \; \inf_{u \in A} L(u, p)$$

Proposition 3
if instead of (3) we have A is either finite or 4(a) holds, then

$$\underset{u \in A}{Min} \; \underset{p \in B}{Sup} \; L(u, p) = \sup_{p \in B} \inf_{u \in A} L(u, p)$$

Proposition 4
if instead of (3) we have B is either finite or 4(b) holds, then

$$\inf_{u \in A} \underset{p \in B}{Sup} \; L(u, p) = \underset{p \in B}{Max} \; \inf_{u \in A} L(u, p)$$

Application to Duality

(P) $\underset{u \in V}{inf} F(u)$ or $\underset{u \in domF}{inf} \; F(u)$
we try to write

$$F(u) = \sup_{p \in B} L(u, p)$$

the Primal problem becomes

$$\inf_{u \in A} \underset{p \in B}{Sup} \; L(u, p)$$

How? we cosider two cases
Case (1) : $F(u) = F_0(u) + F_1(u)$ with $F_1(u)$ proper, lower semicontinuous and convex ($F_1 \in \Gamma_0(V)$)

$$\overset{**}{F_1}(u) = F_1(u) = \sup_{u^* \in \overset{*}{V}} \langle u, u^* \rangle - \overset{*}{F_1}(\overset{*}{u})$$

$$L(u, p) = \langle u, p \rangle - \overset{*}{F_1}(p) + F_0(u)$$

$$F(u) = \sup_{p \in \overset{*}{V}} \langle u, p \rangle - \overset{*}{F_1}(p) + F_0(u)$$

The primal problem becomes

$$\inf_{u\in A}\ \sup_{p\in \overset{*}{V}} \left\{ \langle u,p\rangle - \overset{*}{F_1}(p) + F_0(u) \right\}$$

Case (2) :$F(u) = F_0(u) + F_1(Su)$ with $S : V \longrightarrow Y$ ($\overset{*}{Y} = Z$) S can be nonlinear, $F_1 \in \Gamma_0(V)$

$$F_1(Su) = \sup_{p\in Z} \langle Su,p\rangle - \overset{*}{F_1}(p)$$

$$\langle Su,p\rangle - \overset{*}{F_1}(p) + F_0(u)$$

The primal problem becomes

$$\inf_{u\in V}\sup_{p\in Z} L(u,p)$$

Example: The Mossolev Problem

$$\inf_{u\in H_0^1(\Omega)} \frac{\alpha}{2}\|\nabla u\|^2_{L^2(\Omega)^n} + \beta\|\nabla u\|_{L^1(\Omega)^n} - \langle f,u\rangle$$

$$F(u) = \int_\Omega \left(\frac{\alpha}{2}|\nabla u|^2 + \beta|\nabla u| - fu\right)dx$$

$$S = -\nabla$$

$$F_1(p) = \int_\Omega \left(\frac{\alpha}{2}|p|^2 + \beta|p|\right)dx$$

$$L(u,p) = \int_\Omega \left(-p.\nabla u - \frac{1}{2\alpha}(|p|-\beta)_+^2 - fu\right)dx$$

the primal problem (P)

$$\inf_{u\in H_0^1(\Omega)}\ \sup_{p\in L^2(\Omega)^n} \int_\Omega \left(-p.\nabla u - \frac{1}{2\alpha}(|p|-\beta)_+^2 - fu\right)dx$$

the dual problem $(\overset{*}{P})$

$$\sup_{p\in L^2(\Omega)^n}\ \inf_{u\in H_0^1(\Omega)} \int_\Omega \left(-p.\nabla u - \frac{1}{2\alpha}(|p|-\beta)_+^2 - fu\right)dx$$

(P)

$$\inf_{u\in H_0^1(\Omega)} \int_\Omega (u\,\mathrm{div}\,p - fu)\,dx = \inf_{u\in H_0^1(\Omega)} \int_\Omega (\mathrm{div}\,p - f)\,u\,dx = \begin{cases} 0 & \mathrm{div}\,p - f = 0 \\ -\infty & \text{other wise} \end{cases}$$

$(\overset{*}{P})$ is

$$\sup_{\substack{p\in L^2(\Omega)^n \\ \mathrm{div}\,p = f}} -\frac{1}{2\alpha}\int_\Omega (|p|-\beta)_+^2\,dx$$

Extremality

$$L(\overline{u},\overline{p}) = \inf_{u\in V}\ \sup_{p\in Z} L(u,p) = \sup_{p\in Z}\inf_{u\in V} L(u,p)$$

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$$\int_\Omega -\overline{p}\nabla\overline{u} - \frac{1}{2\alpha}\left(|\overline{p}| - \beta\right)_+^2 - f\ \overline{u}\ dx = \int_\Omega \left(\frac{\alpha}{2}\left|\nabla\overline{u}\right|^2 + \beta\left|\nabla\overline{u}\right| - f\ \overline{u}\right) dx$$

$$\int_\Omega \left(-\overline{p}\nabla\overline{u} - \frac{1}{2\alpha}\left(|\overline{p}| - \beta\right)_+^2 - \frac{\alpha}{2}\left|\nabla\overline{u}\right|^2 - \beta\left|\nabla\overline{u}\right|\right) dx = 0$$

$\Longrightarrow$

$$-\overline{p}\nabla\overline{u} - \frac{1}{2\alpha}\left(|\overline{p}| - \beta\right)_+^2 - \frac{\alpha}{2}\left|\nabla\overline{u}\right|^2 - \beta\left|\nabla\overline{u}\right| = 0$$

$$\nabla\overline{u} = \frac{-\overline{p}}{\alpha\left|\overline{p}\right|}\left(|\overline{p}| - \beta\right)_+$$

$$\operatorname{div}\overline{p} = f$$