Lecture 27

Mossolov's problem (another method of dualization).

The given Problem is
$\inf\limits_{u\in H_0^1(\Omega)} (\frac{\alpha}{2}\|u\|^2_{H_0^1(\Omega)} + \beta\|\nabla u\|_{L^1(\Omega)^n} - \langle f,u\rangle)$ with the following assupmtions:

$V = H_0^1(\Omega),\quad Y = L^2(\Omega)^n,\quad V^* = H_0^{-1}(\Omega),\quad Y^* = Y,$
$A = \nabla,\quad A^* = -\div$ where $\alpha,\beta \gneqq 0$

Note that in the previous lecture this Mossolov's problem was solved with a choice of F and G. Here in this lecture the choice of F and G is different. i.e (another method of dualization).
Now, let $F(u) = -\langle f,u\rangle$.
Then

$$F^*(u^*) = \begin{cases} 0 & \text{if } u^* = -f \\ \infty & \text{other wise} \end{cases}$$

This is done before. (see previous lectures)
$G(p) = \frac{\alpha}{2}\|p\|^2_{L^2(\Omega)^n} + \beta\|p\|_{L^1(\Omega)^n} = \int\limits_{\Omega}(\frac{\alpha}{2}\mid p\mid^2 + \beta\mid p\mid)dx$

we want to find: $G^*(p^*)$. To do so we first start with the following lemma.

Lemma:

Let $g(x) = \frac{\alpha}{2}\mid x\mid^2 + \beta\mid x\mid$ where $g: R^n \to R$, then

$$g^*(y^*) = \frac{1}{2\alpha}(\mid y^*\mid - \beta)^2_+ \qquad \text{where } S_+ = \begin{cases} s & \text{if } s \geq 0 \\ 0 & \text{other wise} \end{cases}$$

and the sup is attained at $\overline{x} = \frac{y}{\alpha|y|}(\mid y\mid - \beta)_+$

Proof:
Let $f(x) = x\cdot y - \frac{\alpha}{2}\mid x\mid^2 - \beta\mid x\mid$ (note that $f: R^n \to R$ and f' is the grad (∇))
then $f'(x) = y - \alpha x - \beta\frac{x}{|x|}$
By setting $f'(x) = 0$ we get: $y = \alpha x + \beta\frac{x}{|x|} = (\alpha + \frac{\beta}{|x|})x$ ......(1)
We want to solve for (x). To do so, multply (1) by x then by y (note: multiplying here means dot product).
Multiplying by x gives:

$$x\cdot y = \alpha\mid x\mid^2 + \beta\mid x\mid = (\alpha\mid x\mid + \beta)\mid x\mid$$

Multiplying by y gives:

$$\begin{aligned} \mid y\mid^2 &= (\alpha + \tfrac{\beta}{|x|})x\cdot y \\ &= (\alpha + \tfrac{\beta}{|x|})(\alpha\mid x\mid + \beta)\mid x\mid) \\ &= (\alpha\mid x\mid + \beta)^2 \end{aligned}$$

$$\Rightarrow \quad \mid y\mid = (\alpha\mid x\mid + \beta)$$

$$\Rightarrow \quad \mid x\mid = \frac{1}{\alpha}(\mid y\mid - \beta)$$

This requires that $\mid y\mid \geq \beta$. Otherewise there are no critical points.

Assume now that $\mid y\mid \geq \beta$.
From (1)

$$\begin{aligned} y &= (\alpha + \frac{\beta}{\frac{1}{\alpha}(|y|-\beta)})x \\ &= (\alpha + \frac{\alpha\beta}{(|y|-\beta)})x \\ &= \frac{\alpha|y|}{|y|-\beta}x \qquad \Rightarrow\ x = \frac{y}{\alpha|y|}(\mid y\mid - \beta) \quad \text{and} \end{aligned}$$

$$f_{\max} = (\alpha\mid x\mid + \beta)\mid x\mid - \frac{\alpha}{2}\mid x\mid^2 - \beta\mid x\mid$$

$$= \frac{\alpha}{2} \mid x \mid^2 = \frac{1}{2\alpha}(\mid y \mid - \beta)^2$$

Note here that for $\mid y \mid \lneqq \beta$, there is no critical values and $f_{\max} = 0$ since

$$\begin{aligned} x \cdot y - \frac{\alpha}{2} \mid x \mid^2 - \beta \mid x \mid \leq \mid x \mid\mid y \mid - \frac{\alpha}{2} \mid x \mid^2 - \beta \mid x \mid \\ \leq -\frac{\alpha}{2} \mid x \mid^2 \\ \leq 0 \end{aligned}$$

Therefore:

$f_{\max} = \frac{1}{2\alpha}(\mid y \mid - \beta)_+^2$ and occurs when $\overline{x} = \frac{y}{\alpha|y|}(\mid y \mid - \beta)_+$

So,

$G^*(p^*) = \frac{1}{2\alpha}\int_\Omega (\mid p^* \mid - \beta)_+^2 dx = \frac{1}{2\alpha}(\|\mid p^* \mid - \beta)_+ \|_{L^2(\Omega)^n}^2$ and

the Dual Problem would be:

$$\text{P}^*: \sup_{p^* \in Y^*} - F(A^* p^*) - G^*(-p^*)$$

$$= \sup_{A^* p^* = -f} \frac{1}{2\alpha}(\|\mid p^* \mid - \beta)_+ \|_{L^2(\Omega)^n}^2$$

note that $A^* p^* = -f$ is closed and convex set

and also $\|\mid p^* \mid - \beta)_+ \|_{L^2(\Omega)^n}^2$ is continouos, coercive, strictly convex which all implies that P* has a unique solution.

***The clear relation between P and P* can be found by using the extramility condition. The relation is given as:** $\nabla \overline{u} = \frac{-\overline{p}^*}{\alpha|\overline{p}^*|}(\mid p^* \mid - \beta)_+$ **and the justification is left as an exercise.**

end of lec#27