Lecture 26 (Mosolev Problem)

$$\begin{array}{llll} V = H_0^1(\Omega) & V^* = H^{-1}(\Omega) & Y = L^1(\Omega)^n & Y^* = L^\infty(\Omega)^n \\ A = \nabla & A^* = -\operatorname{div} & f \in V^* \text{ given} & \alpha, \beta > 0 \end{array}$$

Before we state the problem, we should verify that $A : V \longrightarrow Y$ is continuous. Indeed,

$$A : H_0^1(\Omega) \longrightarrow L^2(\Omega)^n$$

is so. When Ω is finite we have $L^2(\Omega) \subset L^1(\Omega)$ and from Hölder inequality we have

$$\begin{aligned} \int |f| &\leq \sqrt{\int |f|^2}\sqrt{\int 1} \\ \int |f| &\leq C\sqrt{\int |f|^2} \\ \|f\|_1 &\leq C\|f\|_2 \\ \therefore \|\nabla u\|_1 &\leq \|\nabla u\|_2 \\ &\leq k\|u\|_{H_0^1(\Omega)} \end{aligned}$$

So $A : V \longrightarrow Y$ is continuous. The primal problem is

$$\inf_{u \in V} \frac{\alpha}{2}\|u\|_V^2 + \beta\|\nabla u\|_Y - \langle f, u\rangle \left(= \inf_{u \in V} \left\{ \int \frac{\alpha}{2}|\nabla u|^2 + \beta \int |\nabla u| - \int fu \right\}\right)$$

Now, let

$$\begin{aligned} F(u) &= \frac{\alpha}{2}\|u\|_V^2 - \langle f, u\rangle \\ F^*(u^*) &= \frac{1}{2\alpha}\|u^* + f\|_{V^*}^2 \\ G(p) &= \beta\|p\|_Y \end{aligned}$$

To find G^*, let $f(x) = \beta|x| \qquad (x \in I\!R^n)$. Then

$$f^*(y) = \sup_{x \in I\!R^n} x \cdot y - \beta|x|$$

Now let $h(x) = x \cdot y - \beta|x|$, then

$$\begin{array}{l} h'(x) = y - \beta\frac{x}{|x|}, \qquad |x| = \sqrt{x_1^2 + x_2^2 + \cdots x_n^2} \\ h'(x) = 0 \Rightarrow y = \beta\frac{x}{|x|} \Rightarrow |y| = \beta \\ x \cdot y - \beta|x| = x \cdot y - |y||x| \leq |y||x| - |y||x| = 0 \\ x = 0 \qquad \text{or} \qquad x = \gamma y \end{array}$$

So if $|y| = \beta, x = \gamma y$ then $f^*(y) = 0$. If $|y| \neq \beta$ we do not have critical points
<u>case 1:</u> $|y| < \beta$

$$xy - \beta|x| \leq |x||y| - \beta|x| = |x|(|y|\beta) < 0$$

in this case $f^*(y) = 0$ as well.
<u>case 2:</u> $|y| > \beta$
Take $x = \lambda y, \quad \lambda > 0$

$$xy - \beta|x| = \lambda|y|^2 - \beta\lambda|y| = \lambda|y|(|y| - \beta) > 0$$

so

$$f^*(y) = \begin{cases} 0, & |y| \leq \beta \\ \infty, & \text{otherwise} \end{cases}$$

and we have

$$G^*(p^*) = \begin{cases} 0, & |p^*(x)| \le \beta \text{ a.e on } \Omega \\ \infty, & \text{otherwise} \end{cases} = \begin{cases} 0, & \|p^*(x)\|_\infty \le \beta \\ \infty, & \text{otherwise} \end{cases}$$

The dual problem

$$\sup_{p^* \in Y^*} -F^*(A^*p^*) - G^*(p^*) = \sup_{\|p^*(x)\|_\infty \le \beta} -\frac{1}{2\alpha}\| - \nabla \cdot p^* + f\|^2_{V^*}$$

This problem has solutions; because $\|-\nabla p^* + f\|^2_{V^*}$ is convex over a bounded closed convex set $\|p^*(x)\|_\infty \le \beta$.
Extremality conditions

$$\begin{array}{lcllcl} F(\bar{u}) & + & F^*(A^*\bar{p^*}) & = & \langle A^*\bar{p^*} + f, \bar{u}\rangle \\ \frac{\alpha}{2}\|\bar{u}\|^2_V & + & \frac{1}{2\alpha}\|A^*\bar{p^*} + f\|^2_{V^*} & = & \langle A^*\bar{p^*} + f, \bar{u}\rangle \\ \| - \alpha\Delta\bar{u}\|^2_{V^*} & + & \| - \nabla \cdot \bar{p^*} + f\|^2_{V^*} & = & 2\langle -\nabla \cdot \bar{p^*} + f, -\alpha\Delta\bar{u}\rangle \\ -\alpha\Delta\bar{u} & + & \nabla \cdot \bar{p^*} & = & f \end{array}$$