Lecture 23

Dirichlet Problem:

$-\triangle u = f \quad$ on Ω
$u = 0 \quad$ on Γ
$\inf\ (\frac{1}{2}\| \nabla u \|^2 - \langle f, u\rangle)$
$V = H_0^1(\Omega), \quad Y = L^2(\Omega)^n, \quad V^* = H_0^{-1}(\Omega), \quad Y^* = Y$
$F : V \to R$ is defined by $F(n) = -\langle f, u\rangle$

$$F^*(u^*) = \begin{cases} 0 & \text{if } u^* = -f \\ \infty & \text{other wise} \end{cases}$$

$G(p) = \frac{1}{2}\|p\|^2$

$G^*(p^*) = \frac{1}{2}\|p^*\|^2$

Now P is given as:
$\inf\ \frac{1}{2}\| \nabla u \|^2 - \langle f, u\rangle$
where $J(u,p) = \frac{1}{2}\| \nabla u \|^2 - \langle f, u\rangle$ is continouos, coercive (by Poin Care' inequality), and strictly convex which implies that P has a unique solution and P is stable ⇒ P* has a solution and $inf\ P = sup\ P^*$.

Also,
$\Phi(0,p^*) = J^*(A^*p^*, -p^*) = F^*(A^*p^*) + G^*(-p^*)$
⇒P* is given by: $\sup\limits_{p^* \in Y^*} -[F^*(A^*p^*) + G^*(-p^*)] = \sup\limits_{A^*p^*=-f} -G^*(-p^*) = \sup\limits_{A^*p^*=-f} -\frac{1}{2}\|p^*\|^2$
and since $p^* \to \|p^*\|^2$ is continouos, coercive, strictly convex, P* has a unique solution.

Note here that we can find the clear relation between P and P* For the extramility condition as follows:

$F(\overline{u}) + F^*(A^*\overline{p}^*) = \langle \overline{u}, A^*\overline{p}^*\rangle \Rightarrow -\langle f, \overline{u}\rangle = -\langle f, \overline{u}\rangle$ (trivial equation)
and $G(A\overline{u}) + G^*(-\overline{p}^*) = \langle A\overline{u}, \overline{p}^*\rangle \quad \Rightarrow \frac{1}{2}\|\overline{u}^*\|^2 + \frac{1}{2}\|\overline{p}^*\|^2 + \langle \overline{p}^*, \nabla\overline{u}\rangle = 0$
$\Rightarrow \|\nabla\overline{u} + \overline{p}^*\|^2 = 0 \quad \Rightarrow \quad \nabla\overline{u} = -\overline{p}^*$
$\inf\ P = \sup\ P^* = -G^*(-\overline{p}^*) = -\frac{1}{2}\|\overline{p}^*\|^2 = -\frac{1}{2}\|\nabla\overline{u}\|^2$

The nonlinear Dirichlet Problem:
$\inf\ (\frac{1}{\alpha}\| \nabla u \|^\alpha - \langle f, u\rangle)$
with $u \in W_0^{1,\alpha}(\Omega), \quad f \in W_0^{-1,\alpha'}(\Omega), \quad \frac{1}{\alpha} + \frac{1}{\alpha'} = 1 \quad and \quad 1 < \alpha < \infty$

Lemma:
let $f : R \to R$ be defined by $f(x) = \frac{1}{\alpha} \mid x \mid^\alpha$ then
$f^*(y) = \sup\limits_{x\in R} xy - \frac{1}{\alpha} \mid x \mid^\alpha = \frac{1}{\alpha'} \mid y \mid^{\alpha'}$ and the sup occurs at $\overline{x}$
where $\overline{x} \mid \overline{x} \mid^{\alpha-2} = y$

Proof: (EFS)

$V = W^{1,\alpha}(\Omega), \quad Y = L^\alpha(\Omega)^n, \quad Y^* = L^{\alpha'}(\Omega)^n, \quad V^* = W^{-1,\alpha'}(\Omega)$

$F(n) = -\langle f, u\rangle$

$$F^*(u^*) = \begin{cases} 0 & \text{if } u^* = -f \\ \infty & \text{other wise} \end{cases}$$

$G(p) = \frac{1}{\alpha}\|p\|^\alpha_{L^{\alpha'}(\Omega)^n}$

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$G^*(p^*) = \frac{1}{\alpha'}\|p^* \|^{\alpha'}_{L^{\alpha'}(\Omega)^n}$ (to show)

Define: $g(\eta) = \frac{1}{\alpha} \mid \eta \mid^{\alpha} \Rightarrow g^*(\eta) = \sup_{\eta \in Y} \eta \cdot y - g(\eta)$

$$= \sup_{\eta \in Y} \eta \cdot y - \frac{1}{\alpha} \mid \eta \mid^{\alpha}$$
$$= \sup_{\eta \in Y} \eta \cdot y - \frac{1}{\alpha} \sum \mid \eta_i \mid^{\alpha}$$
$$= \sup_{\eta \in Y} \sum \eta_i y_i - \frac{1}{\alpha} \mid \eta_i \mid^{\alpha}$$

and by equating all partial derivative to zero we get:

$$y_i = \mid \eta_i \mid^{\alpha-1} \frac{\eta_i}{\mid \eta_i \mid} = \mid \eta_i \mid^{\alpha-2} \eta_i \quad \Rightarrow$$
$$g^*(y) = \frac{1}{\alpha'} \mid y \mid^{\alpha'}_{\alpha'}$$

i.e. $G^*(p^*) = \frac{1}{\alpha'}\|p^* \|^{\alpha'}_{L^{\alpha'}(\Omega)^n}$ #

and so P* becomes:

$\sup_{A^* p^* = -f} - \frac{1}{\alpha'}\|p^* \|^{\alpha'}_{L^{\alpha'}(\Omega)^n}$ note here as exactly as before (coercivity, strict convexisty...

we have P has unique solution, and P* is so.
end of lec#23