Lecture 14

The Direct Study of Certain Variational Inequalities (continue)

$\langle Au - f, v - u\rangle + \Phi(v) - \Phi(u) \geq 0.\ \ \forall\, v \in V$ where V is a reflexive Banach space, $A:\ V \to V*$, where $f \in V*$ is given and $\Phi : V \to \overline{\overline{R}}$.

i) Φ is proper, lsc and convex.

ii) A is weakly continuous on finite dimensional subspaces of V.

iii) A is a monotone. i.e. $\langle Au - Av, u - v\rangle \geq 0.\ \forall\, u, v \in V$.

iv) A is coercive: $\exists\, v_\circ \in V$ such that: $\dfrac{\langle Av, v - v_\circ\rangle + \Phi(v)}{\| v \|} \to \infty$ as $\| v \| \to \infty$.

Problem:

Find $u \in V$ such that $\langle Au - f, v - u\rangle + \Phi(v) - \Phi(u) \geq 0.\ \ \forall\, v \in V$ (call this *).

Theorem 1 ***Problem (*) has at least one solution.***

Proof:

step (3):

Assume V is of infinite dimension i.e. $dimV = \infty$

Let $\{V_n\}_{n=1}^{\infty}$ be a sequence of FD subspaces of V containing $v_\circ$ that satisfies $\dfrac{\langle Av, v - v_\circ\rangle + \Phi(v)}{\| v \|} \to \infty$ as $\| v \| \to \infty$ where $V_n \subseteq V_{n+1}$ and $\overset{\infty}{\underset{n=1}{\cup}} V_n$. (Note here that having $\{V_n,\ \ n = 1, 2, 3, ...\}$ being just a family of subspaces is not enough to have such $v_\circ$ in all of $V_i,\ \ i = 1, 2, 3, ...$)

Now, for each n $\exists$ a $u_n \in V_n$ s.t.

$\langle Au_n - f, v - u_n\rangle + \Phi(v) - \Phi(u_n) \geq 0.\ \ \forall\, v \in V_n$

and by the discussion made before about the coercivity of A, we have $\{u_n\}$ is bounded.

$\therefore u_n \rightharpoonup u_\circ$ for some $u_\circ \in V$.

Digression to investigate monotonicity:

$\langle Au - Av, u - v\rangle \geq 0$

$\Rightarrow \langle Au, u - v\rangle \geq \langle Av, u - v\rangle$

By putting $u = u_m,\ v = u_\circ,$ we get $\langle Au_m, u_m - u_\circ\rangle \geq \langle Au_\circ, u_m - u_\circ\rangle$, and by taking the $\underline{\lim}$ of both sides as $m \to \infty$,we have(insert a note): $\underline{\lim}\langle Au_m, u_m - u_\circ\rangle \geq 0 \Rightarrow$
$\overline{\lim}\langle Au_m, u_m - u_\circ\rangle \geq 0$ ....(***)

Also, we already have: $\langle Au_n - f, v - u_n\rangle + \Phi(v) - \Phi(u_n) \geq 0.$

so, by fixing n and letting $m \geq n$ we have:

$\langle Au_m - f, v - u_m\rangle + \Phi(v) - \Phi(u_m) \geq 0 \ \ \Rightarrow$

$\Phi(v) - \Phi(u_m) \geq \langle f, v - u_m\rangle + \langle Au_m, u_m - v\rangle$........(****)

Note here that

i) since Φ lsc and convex then $\Phi(u_\circ) = \underset{n\to\infty}{\underline{\lim}}\ \Phi(u_n)$.

ii) $\overline{\lim}(-\Phi(u_n)) = -\underline{\lim}\Phi(u_n)$

iii) $\overline{\lim}(a - \Phi(u_n)) = \overline{\lim}(a + (-\Phi(u_n)) = a + \overline{\lim}(-\Phi(u_n)) = a - \underline{\lim}\Phi(u_n)$

Now taking $\overline{lim}$ of both sides of (****) we get:

$\Phi(v) - \Phi(u_\circ) \geq \langle f, v - u_\circ \rangle + \overline{\lim}\langle Au_m, u_m - v\rangle$.
Note here that $\overline{\lim}$ of LHS of (****) $= \overline{\lim}(\Phi(v) - \Phi(u_m)) = \Phi(v) - \underline{\lim}\Phi(u_m) = \Phi(v) - \Phi(u_\circ)$.
Also, since n is arbitrary, we have the above inequality is true for all n.

Let $v \in V$ and let $v_n \to v$ then
$\Phi(v_n) - \Phi(u_\circ) \geq \langle f, v_n - u_\circ\rangle + \overline{\lim}\langle Au_m, u_m - v_n\rangle$.
Take $\underline{\lim}$ for both side as $n \to \infty$ we get:

$$\begin{aligned}\Phi(v) - \Phi(u_\circ) &\geq \langle f, v - u_\circ\rangle + \underline{\lim}_n \overline{\lim}_m \langle Au_m, u_m - v_n\rangle \\ &\geq \langle f, v - u_\circ\rangle + \overline{\lim}_m \underline{\lim}_n \langle Au_m, u_m - v_n\rangle \\ &= \langle f, v - u_\circ\rangle + \overline{\lim}_m \langle Au_m, u_m - v\rangle \ \ \forall \ v \in V.\end{aligned}$$

Now, if we let $v = u_\circ$ in the above inequality (since it is true $\forall \ v \in V$) we have:

$0 \geq \overline{\lim}_m \langle Au_m, u_m - u_\circ\rangle$

$\Rightarrow \quad 0 \geq \overline{\lim}_m \langle Au_m, u_m - u_\circ\rangle \geq \underline{\lim} \langle Au_m, u_m - u_\circ\rangle \geq 0$ (by(***)above) $\geq \overline{\lim}\langle Au_m, u_m - u_\circ\rangle$.

$\therefore \quad \lim\langle Au_m, u_m - u_\circ\rangle = 0$............(*****)

By going back to monotonicity of A i.e. $\langle Au, u - v\rangle \geq \langle Av, u - v\rangle$ and letting $u = u_m, v = (1-\alpha)u_\circ + \alpha w$ then

$$\begin{aligned}u_m - v &= u_m - (1-\alpha)u_\circ - \alpha w \\ &= u_m - u_\circ + \alpha(u_\circ - w) \\ &= (1-\alpha)(u_m - u_\circ) + \alpha(u_m - w)\end{aligned}$$

and so

$\langle Au_m, (1-\alpha)(u_m - u_\circ) + \alpha(u_m - w)\rangle \geq \langle Av, u_m - u_\circ + \alpha(u_\circ - w)\rangle$

$\Rightarrow \quad (1-\alpha)\langle Au_m, (u_m - u_\circ)\rangle + \alpha(Au_m, u_m - w)\rangle \geq \langle Av, u_m - u_\circ\rangle + \alpha\langle Av, u_\circ - w\rangle$

taking $\underline{\lim}$ for both sides gives:

$\alpha\underline{\lim}\langle(Au_m, u_m - w)\rangle \geq \alpha\underline{\lim}\langle Av, u_\circ - w\rangle$ or:

$\underline{\lim}\langle(Au_m, u_m - w)\rangle \geq \underline{\lim}\langle Av, u_\circ - w\rangle$ and by taking $\lim_{\alpha\to 0}$ $\Rightarrow$ (by using the continuity $v \to u_\circ$ as $\alpha \to 0$ we have $v \rightharpoonup u_\circ$ and so $Av \rightharpoonup Au_\circ$).

$\overline{\lim}\langle Au_m, u_m - w\rangle \geq \underline{\lim}\langle Au_m, u_m - w\rangle \geq \langle Au_\circ, u_\circ - w\rangle \quad \forall\, w \in V$

and by (*****) we have:

$\Phi(w) - \Phi(u_\circ) \geq \langle f, w - u_\circ\rangle + \langle Au_\circ, u_\circ - w\rangle$ or

$\langle Au_\circ - f, w - u_\circ\rangle + \Phi(w) - \Phi(u_\circ) \geq 0$

i.e. it has a solution

Special Cases:

case (1):

$A : C \subseteq V \to V^*$. C is closed and convex. A is a monotone, weakly continuous on a FD subset of C and coercive. Then, there exists a $u \in C$ such that $\langle Au - f, v - u\rangle \geq 0$

proof:

By extending A to the whole space as

$$\overline{A}u = \begin{cases} Au & \text{if} \quad u \in C \\ \infty & \text{if} \quad u \notin C \end{cases}$$

and by using Φ being the indicator function on C, we have the result directly by the previous theorem.

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case (2):

$A : V \to V^*$ with same assumption as above i.e. monotone.,,etc. $\Rightarrow \exists\, u \in V$ s.t. $Au = f$

proof:

By putting $V = C$ in case (1) and letting $v = u + w$ and so $v - u = w$ we get:

$\langle Au - f, w\rangle \geq 0 \quad \forall \; w \in V^* \;\Rightarrow$

$\langle Au - f, -w\rangle \geq 0 \quad \Rightarrow$

$\langle Au - f, w\rangle = 0 \quad \forall \; w \in V^* \;\Rightarrow$

$Au = f$

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case (3):

$A : V \to V^*$ where V is a Hilbert space with $V = V^*.A$ is linear and bounded with $\langle Au, u\rangle \geq \alpha \parallel u \parallel^2$. Then given $f \in V$, there exists a unique $u \in V$ s.t. $Au = f$.

proof:

Note here that a bounded operator is continuous iff it is weakly continuous.

Monotonicity of A:

$$\langle Au - Av, u - v\rangle = \langle A(u - v), u - v\rangle \geq \alpha \parallel u - v \parallel^2 \geq 0$$

Coercivity of A:

$$\frac{\langle Au, u\rangle}{\|u\|} \geq \frac{\alpha\|u\|^2}{\|u\|} = \alpha\|u\| \to \infty \quad \text{as} \quad \|u\| \to \infty$$

So, by case (2) the existence is obtained. The uniqueness of u is obtained easily $\langle Au, u\rangle \geq \alpha \parallel u \parallel^2$
Assume $\exists\, u_1, u_2$ such that $Au_1 = f = Au_2$. Then:

$$\langle Au_1 - Au_2, u_1 - u_2\rangle = \langle A(u_1 - u_2), u_1 - u_2\rangle \geq \alpha \parallel u_1 - u_2 \parallel^2 \Rightarrow$$

$$0 = \langle f - f, u_1 - u_2\rangle \geq \alpha \parallel u_1 - u_2 \parallel^2 \quad \forall\, \alpha \quad \Rightarrow$$

$$0 = u_1 - u_2 \quad \Rightarrow u_1 = u_2$$ i.e. u is unique.
