Lecture 13

The Direct Study of Certain Variational Inequalities

$\langle Au - f, v - u\rangle + \Phi(v) - \Phi(u) \geq 0.\ \forall\, v \in V$ where V is a reflexive Banach space, A: V→V*, where f∈V* is given and $\Phi : V \to \overline{\overline{R}}$.

i) Φ is proper, lsc and convex.

ii) A is weakly continous on finite dimensional subspaces of V.

iii) A is a monotone. i.e. $\langle Au - Av, u - v\rangle \geq 0.\ \forall\, u, v \in V$.

iv) A is coercieve: $\exists\, v_\circ \in V$ such that: $\dfrac{\langle Av, v - v_\circ\rangle + \Phi(v)}{\|v\|} \to \infty$ as $\|\, v \,\| \to \infty$.

Problem:

Find u∈V such that $\langle Au - f, v - u\rangle + \Phi(v) - \Phi(u) \geq 0.\ \forall\, v \in V$ (call this *).

Theorem:

Problem (*) has at least one solution.

Proof:

step (1):

Assume V is finite dimensional (FD) and (dom Φ) is bounded. Also we assume here that V has a Hilbert space structure).

(*) may be rewritten as follows:

$\langle u - (u - Au + f), v - u\rangle + \Phi(v) - \Phi(u) \geq 0.\ \forall\, v \in V$

where $u = Prox_\Phi(u - Au + f)$

Define T:V→dom Φ ⊆ cl(dom Φ) by

Tu: $Prox_\Phi(u - Au + f)$

The idea here is to show that T has a fixed point. If we can show that $Prox_\Phi : V \to$domΦ is continuous then T has a fixed point by Brouwer's fixed point theorem. For that let $f_1, f_2 \in V$, $u_1 = Prox_\Phi f_1$, $u_2 = Prox_\Phi f_2$ then:

$\langle u_1 - f_1, v - u\rangle + \Phi(v) - \Phi(u) \geq 0$

$\langle u_2 - f_2, v - u\rangle + \Phi(v) - \Phi(u) \geq 0$

$\langle u_1 - f_1, u_2 - u_1\rangle + \Phi(u_2) - \Phi(u_1) \geq 0$

$\langle u_2 - f_2, u_1 - u_2\rangle + \Phi(u_1) - \Phi(u_2) \geq 0$ by summing the last two inequalities we get:

$\langle (u_1 - f_1) - (u_2 - f_2), u_2 - u_1\rangle \geq 0$ or by rearranging:

$\langle (u_1 - u_2) - (f_1 - f_2), u_2 - u_1\rangle \geq 0 \implies$

$\|u_2 - u_1\|^2 \leq -\langle f_1 - f_2, u_2 - u_1\rangle \leq \|f_1 - f_2\|\|u_2 - u_1\| \implies \|u_2 - u_1\| \leq \|f_1 - f_2\|$

Therefore it is continous and so T has a fixed point u ∈ cl(domΦ) and because u = Tu ∈ domΦ since range T is in domΦ

∴ (*) has a solution.

Step (2):

Now assume only that V is FD.

For n = 1,2,3,....., define $\Phi_n(u) = \begin{cases} \Phi(u) & \text{if } \|\, u \,\| \leq n \\ \infty & \text{if } \|\, u \,\| \gneqq n \end{cases}$

Note that dom$\Phi_n \subseteq \overline{B(0.n)}$.

By step (1) the problem $\langle Au - f, v - u\rangle + \Phi_n(v) - \Phi_n(u) \geq 0$ has a solution $u_n \in$ dom$\Phi_n \subseteq \overline{B(0.n)}$

i.e. $\langle Au_n - f, v - u_n\rangle + \Phi_n(v) - \Phi_n(u_n) \geq 0.\ \forall\, v \in V$.

Now calaim that $\{u_n\}$ is bounded. If we assume not then we have:

$\langle Au_n - f, v_\circ - u_n\rangle + \Phi_n(v_\circ) - \Phi(u_n) \geq 0$ (note here that $\Phi_n(u_n) = \Phi(u_n)$ since $\|u_n\| \leq n$)

$\Rightarrow \langle Au_n, u_n - v_\circ\rangle + \Phi(u_n) \leq \Phi_n(v_\circ) - \langle f, v_\circ - u_n\rangle$

note here that for sufficiently large $n \geq \|v_\circ\|$, we have $\Phi_n(v_\circ) = \Phi(v_\circ)$ and so

$\langle Au_n, u_n - v_\circ\rangle + \Phi(u_n) \leq \Phi(v_\circ) - \langle f, v_\circ - u_n\rangle$ and by dividing every thing by $\|u_n\|$ we get:

$\dfrac{\langle Au_n, u_n - v_\circ\rangle + \Phi(u_n)}{\|u_n\,\|} \leq \dfrac{\Phi(v_\circ)}{\|u_n\,\|} + \|f\,\|\,(1 + \dfrac{\|v_\circ\,\|}{\|u_n\,\|})$ which $\to \|f\,\| \lneqq \infty$ as $\|u_n\,\| \to \infty$ and this of course conradicts the coercevity. Hence, $\{u_n\}$ is bounded.

Now, since $\{u_n\}$ is bounded in a FD space, there exists a subsequence $\{u_{n_j}\}$ and a $u \in V$ such that

$u_{n_j} \to u$ and ($A_{u_j} \to A_u$ by continuity of A).
Letting $v \in V \Rightarrow \langle Au_{n_j} - f, v - u_{n_j} \rangle + \Phi_{n_j}(v) - \Phi(u_{n_j}) \geq 0$
Then for sufficiently large n_j with $\|v\| \leq n_j$ we have $\Phi_{n_j}(v) = \Phi(v)$
$\therefore$ taking the limit of both sides as $j \to \infty$ we get
$\langle Au - f, v - u \rangle + \Phi(v) - \Phi(u) \geq 0$ and this completes the proof.

Remark :

If $Au_n \to Au$ then $\langle Au_n, u \rangle \to \langle Au, u \rangle$ but it not always true that $\langle Au_n, u_n \rangle \to \langle Au, u \rangle$ whenever $u_n \to u$.Acutually
this can not happen unless we impose the conition of boundedness on either Au_n or u_n. Note on the following:
$\langle Au_n, u_n \rangle = \langle Au_n, u - u + u_n \rangle = \langle Au_n, u \rangle + \langle Au_n, u_n - u \rangle \to \langle Au, u \rangle + \langle Au_n, u_n - u \rangle$
But $| \langle Au_n, u_n - u \rangle | \leq \| Au_n \| \| u_n - u \|$ .....(**)
And since $\| u_n - u \| \to 0$ as $u_n \to u$ then the r.h.s of (**) will not vanished unless $\| Au_n \|$ is bounded.
Similar argument can be done on Au_n to have u_n being bouned.