Lemma F is lsc iff for any $c \in \mathbb{R}, \{u \in V : F(u) \leq c\}$ is closed.

Proof. Assume F is lsc. Let $c \in \mathbb{R}, K = \{u \in V : F(u) \leq c\}$ and $\overline{u} \in \overline{K}$. Take a net $\{u_\gamma\}_{\gamma \in \Gamma}$ such that $u_\gamma \to \overline{u}$. Then

$$F(u_\gamma) \leq c \ \forall \gamma \in \Gamma.$$

Taking the lim inf,

$$F(\overline{u}) \leq \varliminf_{u_\gamma \to \overline{u}} F(u_\gamma) \leq c \ .$$

Therefore, $\overline{u} \in K$. On the other hand, suppose theat or any $c \in \mathbb{R}, \{u \in V : F(u) \leq c\}$ is closed. Let $\overline{u} \in V$ and let $\{u_\gamma\}_{\gamma \in \Gamma}$ be a net such that $u_\gamma \to \overline{u}$. Let $c = \varliminf_{u_\gamma \to \overline{u}} F(u_\gamma)$. If $c \in \mathbb{R}$, then we can extract a subnet $\{u_{\gamma_n}\}$of $\{u_\gamma\}_{\gamma \in \Gamma}$ such that $F(u_{\gamma_n}) \leq c + \frac{1}{n}$. Set $K_n = \left\{u \in V : F(u) \leq c + \frac{1}{n}\right\}$ and $K = \{u \in V : F(u) \leq c\}$. Then $K = \cap_{n=1}^{\infty} K_n$. Since $u_{\gamma_n} \to \overline{u}$ and $u_{\gamma_n} \in K_n$, $\overline{u} \in K$. Thus $F(\overline{u}) \leq c$. If $c = \infty$, there is nothing to prove. If $c = -\infty$, we can repeat the same argument with $K_n = \{u \in V : F(u) \leq -n\}, K = \{u \in V : F(u) = -\infty\}$. ■

Cor 2.4 Suppose V is normed, $F : V \to \overline{\mathbb{R}}$ is proper and conves. TFAE

(i) $\exists$ an open set $\mathcal{O} \subseteq V$ on which F is bounded above.

(ii) $\overbrace{\operatorname{dom} F}^{\circ} \neq \phi$ and F is locally Lipchitz there

Proof. We show only (i) $\Rightarrow$ (ii). The fact that $\overbrace{\operatorname{dom} F}^{\circ} \neq \phi$ follows from Proposition 3.1. It follows also from the same proposition that F is continuous on $\overbrace{\operatorname{dom} F}^{\circ}$. Let $u \in \overbrace{\operatorname{dom} F}^{\circ}$. Since F is continuous at u. it is absolutely bounded, say by a in a ball $\overline{B(u,r)} \subset \overbrace{\operatorname{dom} F}^{\circ}$. Let $v \in B(u,r)$. Let w_1, w_2 be the two ends of the diagonal through u, v and suppose $v \in (u, w_1)$. Write $v = (1-\lambda)u + \lambda w_1$. Then

$$\lambda = \frac{\|v-u\|}{\|w_1 - u\|} = \frac{\|v-u\|}{r}$$

$$\begin{aligned} F(v) - F(u) &= F((1-\lambda)u + \lambda w_1) - F(u) \\ &\leq \lambda(F(w_1) - F(u)) = (F(w_1) - F(u)) \frac{\|v-u\|}{r} \\ &\leq \frac{a - F(u)}{r} \|v-u\| \leq \frac{2a}{r} \|v-u\| . \end{aligned}$$

Let v_1 be the point in $B(u,r)$ which is diagonally opposite of v. Then $u = \frac{1}{2}(v + v_1)$ and

$$F(u) \leq \frac{1}{2}(F(v) + F(v_1)) .$$

Hence,

$$F(u) - F(v) \leq F(v_1) - F(u) .$$

Furthermore, $v_1 = (1-\lambda)\,u + \lambda w_2$, where $\lambda = \frac{\|v_1 - u\|}{\|w_2 - u\|} = \frac{\|v-u\|}{r}$. This yields as before,

$$\begin{aligned} F(u) - F(v) &\leq \lambda\left(F(w_2) - F(u)\right) \\ &\leq \frac{a - F(u)}{r}\,\|v-u\| \leq \frac{2a}{r}\,\|v-u\|. \end{aligned}$$

Therefore,

$$|F(u) - F(v)| \leq \frac{2a}{r}\,\|v-u\|,$$

which proves the Lipschitz continuity at u. For arbitrary $u, v \in \overbrace{\operatorname{dom} F}^{\circ}$, cover the segment $[u,v]$ by a finite number of balls $\{B(u_i, r_i)\}_{i=1}^n$ and observe that $\|v - u_i\| = c_i\,\|v-u\|$. Then

$$|F(u) - F(v)| \leq \sum_{i=1}^{n} \frac{2a_i}{r_i}\,\|v_i - u\| = \left(\sum_{i=1}^{n} \frac{2a_i}{r_i} c_i\right)\|v-u\|.$$

■