II.2 Wavelet Transform Modulus Maxima

April 12, 2009

1 Maxima Lines

We know that the decay rate of the wavelet transform of an α-regular signal f in at a point v is governed by the relation

$$|W_\psi f\,(a,b)| \le A a^{\alpha+1/2}\left(1+\left|\frac{b-v}{a}\right|^{\alpha}\right)\ \forall\,(a,b)\in\mathbb{R}^{+}\times\mathbb{R}.$$

and that if the decay rate, with respect to the scale a is slightly larger than α then f is Lipα. We will see in this section that it is not necessary to measure this decay rate in whole neighborhoods of v. In fact, it suffices to consider only the points where $|W_\psi f\,(a,b)|$ acheives local maxima. Such local maxima occure when

$$\frac{\partial W_\psi f\,(a,b)}{\partial b}=0. \tag{1}$$

Any solution curve $a=a\,(b)$ of equation (1) is called a *maxima line.*

Suppose $f\in L^2(\mathbb{R})$ and $\psi\in C_b^n(\mathbb{R})$ with compact support and $n\ge 0$. Then f is locally integrable on $\mathbb{R}$ and, since $W_\psi f\,(a,b)=\langle f,\psi_{a,b}\rangle$,

$$\frac{\partial^k W_\psi f\,(a,b)}{\partial b^k}=\frac{\partial^k}{\partial b^k}\langle f,\psi_{a,b}\rangle=\left\langle f,\frac{\partial^k}{\partial b^k}\psi_{a,b}\right\rangle=-\frac{1}{a^k}\left\langle f,\psi_{a,b}^{(k)}\right\rangle,\ 0\le k\le n,$$

which shows that the wavelet transform $W_\psi f\,(a,b)$ of a function f by a smooth wavelet ψ is smooth, in the that $W_\psi f\,(a,\cdot)$ is differentiable, even if the original function was not smooth. The implicit function theorem guaratees that if (a_0,b_0) is a solution of (1) such that $\frac{\partial^2 W_\psi f(a_0,b_0)}{\partial b^2}\ne 0$ then there is a unique maxima line, defined in a neighborhood of (a_0,b_0) and passing through (a_0,b_0).

2 Isolated Singualrities

We have seen before that if f has oscillating singularity at a point v then we can find maxima lines outside the cone of influence C_v of v. In this section we investigate another type of singularities of f, namely, isolated singularities.

Definition 1 *(isolated singularities)*

A function f is said to have an isolated *α-singulrity at a point $v \in \mathbb{R}$ if there exists an $\varepsilon > 0$ such that f is Lip$(\alpha + 1)$ at every point in $(v - \varepsilon, v + \varepsilon)$ except v itself where it is Lipα only.*

To discuss the isolated singularities of a function f we need the to state the reguarity of f in terms of its wavelet transform modulus maxima. The following theorem provides sufficient conditions for a function to be locally Lip regular.

Theorem 2 *(local regularity of functions)*

Suppose $\psi \in C_b^n(\mathbb{R})$ has compact support in $[-C, C]$ and n vanishing moments. Let f be locally integrable on $\mathbb{R}$. If $|W_\psi f(a, b)|$ has no maximum in $[\alpha, \beta] \times (0, a_0)$ then f is uniformly Lipn on any closed subinterval of (α, β).

Theorm 2 implies that if f has an n-singularity with $\alpha \leq n$ at a point v then there exists a sequence $(a_k, b_k) \to (0, v)$ such that $|W_\psi f(a_k, b_k)|$ is a local maximum. In other words, there exists a a sequence $(a_k, b_k) \to (v, 0)$ such that

$$\frac{\partial W_\psi f(a_k, b_k)}{\partial b} = 0.$$

This is true, in particular, if a maxima line converges to $(0, v)$.

We have seen that if a signal f has an oscillating singularity at a point v then the modulus maxima lines converging to $(0, v)$ lie outside any cone with vertex at v. Consequently, if f has an isolated singularity at a point v that is not oscillatory, then all modulus maxima lines converging to $(0, v)$ are contained in a cone $|b - v| \leq Da$ for sufficiently small α. In other words, there exists a $D > 0$ and an $a_0 > 0$ such that for all $a \leq a_0$ all modulus maxima lines converging to $(0, v)$ are contained in the cone $|b - v| \leq Da$. In this case the α-regularity of f at v is determined by following the modulus maxima lines converging to $(0, v)$. Since on any of these lines we must have

$$|W_\psi f(a, b)| \leq A a^{\alpha + 1/2},$$

or

$$\log |W_\psi f\ (a,b)| \leq \log A + (\alpha + 1/2) \log a$$

$(\alpha + 1/2)$ is the maximum slope of $\log |W_\psi f\ (a,b)|$ as a function of $\log a$ along the maxima lines converging to $(0, v)$.

Assignment 2 **(a)** Give examples of functions which have isolated α-singularities at $t = 1$ where $\alpha = 1, 2, 3$.

(b) Produce the scalograms for your example functions when analyzed with the wavelets gaus1, gaus2, gaus 3 and gaus4.

(c) Plot the wavelet transform modulus maxima in each case.

(d) Plot the graphs of $\log |W_\psi f\ (a,b)|$ vs $\log a$ and identify the α-regularity of your examples