# II Wavelet Zoom

March 31, 2009

## 1 Lipschitz Regularity

**Definition 1** *(Lipschitz regularity of functions)*
*Suppose $f$ is a function defined on $\mathbb{R}$, $\alpha \geq 0$.*

**(i)** *$f$ is said to be Lipschitz $\alpha$ (or Lip$\alpha$ for short) at a point $v \in \mathbb{R}$ if there exists a polynomial $p_v$ of degree $m := \lfloor \alpha \rfloor$ and a constant $K = K(v) > 0$ such that*

$$|f(t) - p_v(t)| \leq K |t - v|^{\alpha} \ \ \forall t \in \mathbb{R}. \tag{1}$$

**(ii)** *$f$ is said to be uniformly Lip$\alpha$ on $[a, b] \subset \mathbb{R}$ if it is Lip$\alpha$ at every point $v \in [a, b]$ with the constant $K$ independent of $v$.*

**(iii)** *The Lipschitz regularity of $f$ at $v$ is the sup of all $\alpha$ such that $f$ is Lip$\alpha$ at $v$.*

If $f$ is Lip0 at $v$ then $f$ is bounded but may be discontinuous. If $f$ is Lip$\alpha$ but not Lip$\alpha + 1$ at $v$ we say that $f$ has a $\alpha$ singularity at $v$. The Lipschitz regularity of $f$ may vary from one $v$ to the other. In other words, Lipschitz regularity is a local property of the function $f$. One may construct multifractal functions $f$ with distinct Lipschitz regularity at distinct points. In contrast, uniform Lipschitz reguarity expresses a global propery of $f$. The following lemma gives some immediate consequences of Definition 1.

**Lemma 2** *(properties of Lipschitz regularity)*

1. *If $\alpha > m$ and $f$ is Lip$\alpha$ at $v$ then the polynomial $p_v$ is unique.*

2. *If $\alpha > m$, $f$ is Lip$\alpha$ and $f$ is $m$ times continuously differentiable at $v$ then $p_v$ is the Taylor polynomial of degree $m$ at $v$:*

$$p_v(t) = \sum_{k=0}^{m} \frac{f^{(k)}(v)}{k!}(t-v)^k.$$

3. *f $\alpha > m$, $f$ is Lip$\alpha$ and $f$ is $m$ times continuously differentiable at $v$ then $f^{(k)}$ is Lip$(\alpha - k)$ at $v$ for all $k \leq m$.*

4. *If $\alpha > m$ and $f$ is uniformly Lip$\alpha$ on $[a,b]$ then $f$ is $m$ times continuously differentiable on $(a,b)$.*

**Excercise 1** Let $f(t) = \begin{cases} t^2, & |t| < 1 \\ |t-1|, & |t| \geq 1 \end{cases}$. Show that $f$ is Lip$\alpha$ at 0 if and only if $\alpha \in (1,2]$.

**Ecercise 2** Let $f(t) = \begin{cases} |t-1|, & |t| < 1 \\ (|t|-2)^2, & |t| \in [1,3] \\ 0, & |t| > 3 \end{cases}$. Discuss the $\alpha$-regularity of $f$ at 0, 1 and 2.

**Excercise 3** Give an example of a function $f$ which is not Lip$\alpha$ for any $\alpha \geq 0$. Justify your answer.

The Fourier transform can provide information on the global regularity of a function $f$ as in the following theorem.

**Theorem 3** *(Fourier Transform and global reguarity)*
*A function $f$ is bounded and uniformly Lip$\alpha$ on $\mathbb{R}$ if*

$$\int_{-\infty}^{\infty} \left|\widehat{f}(\omega)\right| (1 + |\omega|^\alpha)\, d\omega < \infty.$$

## 2 Vanishing moments and rate of decay

It is clear from Theorem 3 that the local regularity of a function $f$ at a particular point $v$ cannot be determined from the decay rate of its Fourier transform. This can more efficiently be done with a wavelet transform. To study the relation between the regularity of a function $f$ and the decay rate of its wavelet transform we need to introduce the notion of vanishing moments and rate of decay. Let us denote by $C_b^n(\mathbb{R})$ the space of functions that are continuous and bounded together with their first $n$ derivatives.

**Definition 4** *(vanishing moments)*

**(i)** *A function $f \in C_b^n(\mathbb{R})$ is said to have $n \in \mathbb{N}$ vanishing moments if*

$$\int_{-\infty}^{\infty} t^k f(t)\, dt = 0 \text{ for } 0 \leq k < n.$$

**(ii)** *A function $f \in C_b^n(\mathbb{R})$ is said to have decay rate $m \in \mathbb{N}$ if there exists a constant $C$ such that*

$$\left|f^{(k)}(t)\right| \leq \frac{C_m}{1+|t|^m},\ 0 \leq k \leq n,\ \forall t \in \mathbb{R}.$$

**(iii)** *A function $f \in C_b^{\infty}(\mathbb{R})$ is said to have fast decay if $f$ is infinitely differentiable on $\mathbb{R}$ and for all $k, l \in \mathbb{N}$*

$$\left|t^k f^{(l)}(t)\right| \leq C_{kl}\ \forall t \in \mathbb{R}.$$

It follows from the definition that a function $f$ with $n$ vanishing moments is "orthogonal" to all polynomials of degree less than $n$. A function has fast decay if and only if it has decay rate $m$ for all $m \in \mathbb{N}$.

**Lemma 5** *(vanishing moments of translated and dilated wavelets)*
*If $\psi \in C_b^n(\mathbb{R})$ is a wavelet with $n$ vanishing moments then for all $(a,b) \in \mathbb{R}^+ \times \mathbb{R}$*

$$\int_{-\infty}^{\infty} (t-b)^k \psi_{a,b}(t)\, dt = 0 \text{ for } 0 \leq k < n.$$

The following facts from advanced calculus will be useful.

- If $f \in C_b^k(\mathbb{R})$ and $f^{(j)}(0) = 0,\ 0 \le j < k$ then there exists a function $g \in C_b(\mathbb{R})$ such that $f(t) = t^k g(t)$ for every $t \in \mathbb{R}$.
- If a function $f \in C_b(\mathbb{R})$ has decay rate $m$ then $\widehat{f} \in C_b^{m-2}(\mathbb{R})$.

**Theorem 6** *(characterization of wavelets with vanishing moments)* *Assume $m$ and $n$ are integers with $m \ge n+2$.*

**(a)** *A wavelet $\psi \in C_b^n(\mathbb{R})$ has $n$ vanishing moments and decay rate $m$ if and only if there exists a function $\theta \in C_b^{2n}(\mathbb{R})$ with decay rate $m$ such that*

$$\psi = (-2\pi i)^n \frac{d^n \theta}{dt^n}. \tag{2}$$

**(b)** *Moreover, $\psi$ has no more than $n$ vanishing moments if and only if $\int_{-\infty}^{\infty} \theta(t)\, dt \ne 0$.*

**Proof.** Since $\psi$ has decay rate $m \ge n+2$,

$$\int_{-\infty}^{\infty} |\psi(-t)| \left(1 + |t|^{m-2}\right) dt \le C_m \int_{-\infty}^{\infty} \frac{1+|t|^{m-2}}{1+|t|^m} dt < \infty.$$

Since $\mathcal{F}\widehat{\psi} = \mathcal{R}\psi$, it follows from Theorem 3 that $\widehat{\psi} \in C_b^{m-2}(\mathbb{R}) \subseteq C_b^n(\mathbb{R})$. Furthermore, since $\psi$ has $n$ vanishing moments and $\mathcal{F}\left(t^k \psi\right)(\omega) = \left(\frac{1}{2\pi i}\right)^k \widehat{\psi}^{(k)}(\omega)$,

$$0 = \int_{-\infty}^{\infty} t^k \psi(t)\, dt = \left(\frac{1}{2\pi i}\right)^k \widehat{\psi}^{(k)}(0),\ 0 \le k < n.$$

Thus, $\widehat{\psi} \in C_b^n(\mathbb{R})$ and $\widehat{\psi}^{(k)}(0),\ 0 \le k < n$. Hence, there exists a function $\widehat{\theta} \in C_b(\mathbb{R})$ such that

$$\widehat{\psi}(\omega) = \omega^n \widehat{\theta}(\omega) = \left(-\frac{1}{2\pi i}\right)^n \mathcal{F}\theta^{(n)}(\omega). \tag{3}$$

This verifies (2). The decay rate of $\theta$ can be shown by induction on $n$. For $n = 1$, $\widehat{\psi}(\omega) = \omega \widehat{\theta}(\omega) = -\frac{1}{2\pi i} \mathcal{F}\theta'(\omega)$; thus

$$\theta'(t) = -\frac{1}{2\pi i} \psi(t)$$

and

$$\left|\theta'(t)\right| \leq \frac{C_m/2\pi}{1+|t|^m}.$$

Also, we may write

$$\theta(t) = -\frac{1}{2\pi i}\int_{-\infty}^{t}\psi(s)\,ds = -\frac{1}{2\pi i}\int_{t}^{\infty}\psi(s)\,ds.$$

Then, for $t \geq 0$

$$\frac{\int_t^\infty |\psi(s)|\,ds}{1+t^m} \leq C_m \frac{\int_t^\infty \frac{1}{1+s^m}ds}{1+t^m}.$$

The expression on the right is a continuous fucntion which has finite value at $t = 0$ since $m > 2$ and tends to zero as $t \to \infty$ . This means that this function is bounded for all $t \geq 0$. It follows that

$$|\theta(t)| \leq \frac{C_m}{2\pi}\frac{\int_t^\infty \frac{1}{1+s^m}ds}{1+t^m} \leq K_m.$$

A similar estimate can also be established for $t < 0$. Thus $\theta$ has decay rate $m$ for $n = 1$. The rest of the induction steps are straighforward.

Conversely, if (2) for some $\theta \in C_b^{2n}(\mathbb{R})$ with decay rate $m$ then (3) holds which shows that $\widehat{\psi}^{(k)}(0) = 0$ and thus, $\psi$ has $n$ vanishing moments. Furthermore,

$$\left|\psi^{(k)}(t)\right| = (2\pi)^n \left|\theta^{(n+k)}(t)\right| \leq \frac{C_m}{1+|t|^m},\ 0 \leq k \leq n,\ \forall t \in \mathbb{R}.$$

To prove (b) we compute

$$\begin{aligned}\int_{-\infty}^{\infty} t^n\psi(t)\,dt &= (-2\pi i)^n\int_{-\infty}^{\infty} t^n\frac{d^n\theta}{dt^n}dt \\ &= (-2\pi i)^n\left[\left. t^n\frac{d^{n-1}\theta}{dt^{n-1}}\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} t^{n-1}\frac{d^{n-1}\theta}{dt^{n-1}}dt\right].\end{aligned}$$

The boundary term vanishes since $m > n$ and

$$\left|t^n\frac{d^{n-1}\theta}{dt^{n-1}}\right| \leq K_m\frac{|t|^n}{1+|t|^m} \to 0 \text{ as } |t| \to \infty.$$

Using induction we get

$$\begin{aligned}\int_{-\infty}^{\infty} t^n \psi(t)\, dt &= -(-2\pi i)^n \int_{-\infty}^{\infty} t^{n-1} \frac{d^{n-1}\theta}{dt^{n-1}} dt \\ &= \dots \\ &= (2\pi i)^n \int_{-\infty}^{\infty} \theta(t)\, dt.\end{aligned}$$

Hence, $\int_{-\infty}^{\infty} t^n \psi(t)\, dt \neq 0$ if and only if $\int_{-\infty}^{\infty} \theta(t)\, dt \neq 0$. ■

**Corollary 7** *(vanishing moments of fast decaying wavelets)*

*A wavelet $\psi$ has fast decay and $n$ vanishing moments if and only if there exists a function $\theta$ with fast decay such that (2) is satisfied. Moreover, $\psi$ has no more than $n$ vanishing moments if and only if $\int_{-\infty}^{\infty} \theta(t)\, dt \neq 0$.*

# 3 Regularity Measurements with Wavelets

**Theorem 8** *(wavelet measurement of regularity)*

*Suppose a wavelet $\psi \in C_b^n(\mathbb{R})$ has $n$ vanishing moments and decay rate $m$. If $f \in L^2(\mathbb{R})$ is uniformly Lip$\alpha$ with $\alpha \leq n$ at $v$ then there exists $A > 0$ such that*

$$|W_\psi f(a,b)| \leq A a^{\alpha+1/2} \left(1 + \left|\frac{b-v}{a}\right|^{\alpha}\right) \ \forall (a,b) \in \mathbb{R}^+ \times \mathbb{R}. \tag{4}$$

*Conversely, if $\alpha < n$ is not an integer and there exists a constant $A > 0$ and $\alpha' < \alpha$ such that*

$$|W_\psi f(a,b)| \leq A a^{\alpha+1/2} \left(1 + \left|\frac{b-v}{a}\right|^{\alpha'}\right) \ \forall (a,b) \in \mathbb{R}^+ \times \mathbb{R} \tag{5}$$

*then $f$ is Lip$\alpha$ at $v$.*

**Corollary 9** *(wavelet measurment of uniform regularity)*

*Suppose a wavelet $\psi \in C_b^n(\mathbb{R})$ has $n$ vanishing moments and decay rate $m$. If $f \in L^2(\mathbb{R})$ is uniformly Lip$\alpha$ with $\alpha \leq n$ over $I = [b_1, b_2]$ then there exists a constant $A > 0$ such that*

$$|W_\psi f(a,b)| \leq A a^{\alpha+1/2} \ \forall (a,b) \in \mathbb{R}^+ \times I. \tag{6}$$

*Conversely, suppose that $f$ is bounded and $W_\psi f$ satisfies (6) for an $\alpha < n$ that is not an integer. Then $f$ is uniformly Lip$\alpha$ on $I_\epsilon := [b_1 + \epsilon, b_2 - \epsilon]$ for any $\epsilon > 0$.*

**Cone of influence** If a wavelet $\psi$ is supported in $[-C, C]$ then $\psi_{a,b}$ is supported in $[b - aC, b + aC]$. Any local behavior of a signal $f$ at an instant $v$ in its domain will be "reported" by all wavelets $\psi_{a,b}$ for which $v \in [b - aC, b + aC]$ or

$$\left|\frac{b - v}{a}\right| \leq C. \tag{7}$$

This leads us to the following definition.

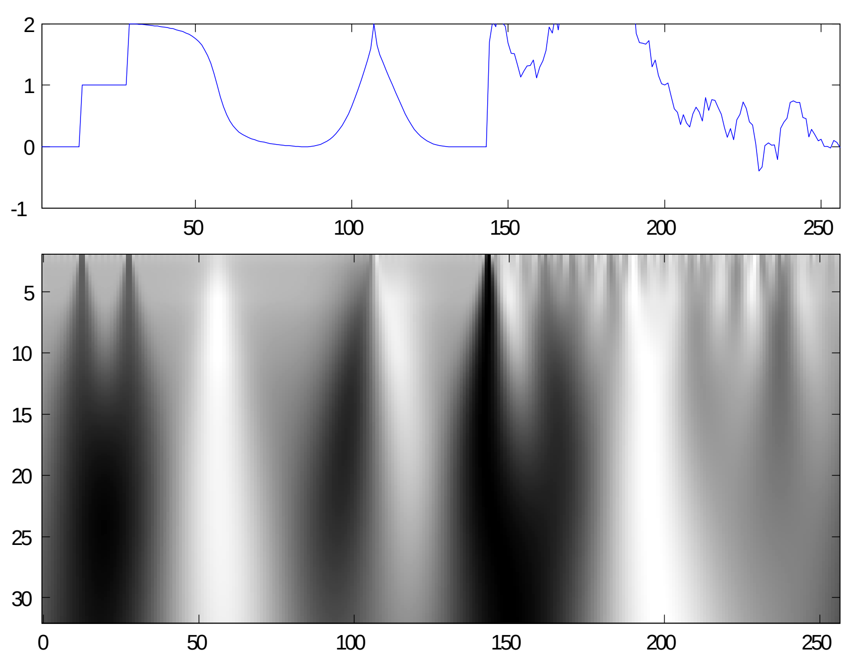
**Definition 10** *(cone of influence of a point)*

*The cone of influence of a point $v$ is the set of points $(a, b)$ such that (7) is satisfied.*

If the signal $f$ is Lip$\alpha$ at $v$ and $(a, b)$ is in the cone of influence of $v$ then equations (4), (5) take the form

$$|W_\psi f\,(a, b)| \leq A' a^{\alpha+1/2}.$$

On the other hand if $f$ has a singularity at $v$ then its cone of influence produces high amplitude coefficients $|W_\psi f\,(a, b)|$. See the figure below



### 3.1 Application to Detecting Oscillating Singularities

The wavelet transform amplitude of a Lip$\alpha$ signal continue to be controlled outside the cone of influence. Indeed if a signal $f$ is Lip$\alpha$ at a point $v$ then for all $(a,b)$ outside the cone of influence, that is, for all $|b-v|>aC$, we get

$$\left|\frac{b-v}{Ca}\right|^{\alpha'}>1.$$

Then equations (4), (5) take the form

$$|W_{\psi}f\,(a,b)|\leq A'a^{\alpha-\alpha'+1/2}\,|b-v|^{\alpha'}.$$

This behavior outside the cone of influence is necessary to determine for sure that the analysed function is Lip$\alpha$. The following lemma shows that if we restrict our attention to the cone of influence and analyze with a smooth wavelet we may get the false impression that the analyzed function is smooth.

**Lemma 11** *Suppose the analyzing wavelet $\psi\in C_b^2(\mathbb{R})$ has two vanishing moments, fast decay and support in $[-C,C]$. Let $f$ be Lip$0$ in a neighborhood of $v=0$. Then*

$$|W_{\psi}f\,(a,b)|\leq Aa^{5/2}\tag{8}$$

*for all sufficiently small $a,b$ in the cone of influence of $0$. In particular, if $f$ is a bounded function, then (8) holds for all $(a,b)$ in the cone of influence of $0$.*

**Proof.** Assume $f$ is bounded, say by $M$ in the neighborhood $(-\epsilon,\epsilon)$ of $0$. Using Theorem 6 we may write

$$\psi\,(t)=\theta''\,(t)\,.$$

Then

$$D_a\psi=a^2\,(D_a\theta)''$$

and

$$|W_{\psi}f\,(a,b)|=|\langle f,D_{a,b}\psi\rangle|=a^2\left|\int_{b-aC}^{b+aC}f\,(t)\,(D_a\theta)''\,(t-b)\,dt\right|$$

For sufficiently small $a, b$ the we have $0 \in [b - aC, b + aC] \subset (-\epsilon, \epsilon)$ and $(a, b)$ is in the cone of influence of 0. This is the case, for example, if $aC < \epsilon$ and $|b| < \min\{aC, \epsilon - aC\}$. Then

$$\begin{aligned}
|W_\psi f(a,b)| &\leq Ma^2 \int_{b-aC}^{b+aC} \left|(D_a\theta)'(t-b)\right| dt \\
&= Ma^2 \int_{-aC}^{aC} \left|(D_a\theta)''(t)\right| dt = Ma^{3/2} \int_{-aC}^{aC} |g''(t)|\, dt \\
&\leq 2MNCa^{5/2} = Aa^{5/2}.
\end{aligned}$$

where $g(t) = \theta\left(\frac{t}{a}\right)$ and $N$ is the maximum of $g''$ in the interval $[-aC, aC]$.
■

Signals $f$ with singularity at $v$ may then produce small amplitude wavelet coefficients inside the cone of influence, but then high amplitude coefficients will be produced outside the cone of influence. As an example we will show that signals $f$ with oscillating singularities do produce high amplitude coefficients outside the cone of influence. For simplicity we will take the model case

$$f(t) = \sin\frac{1}{t}$$

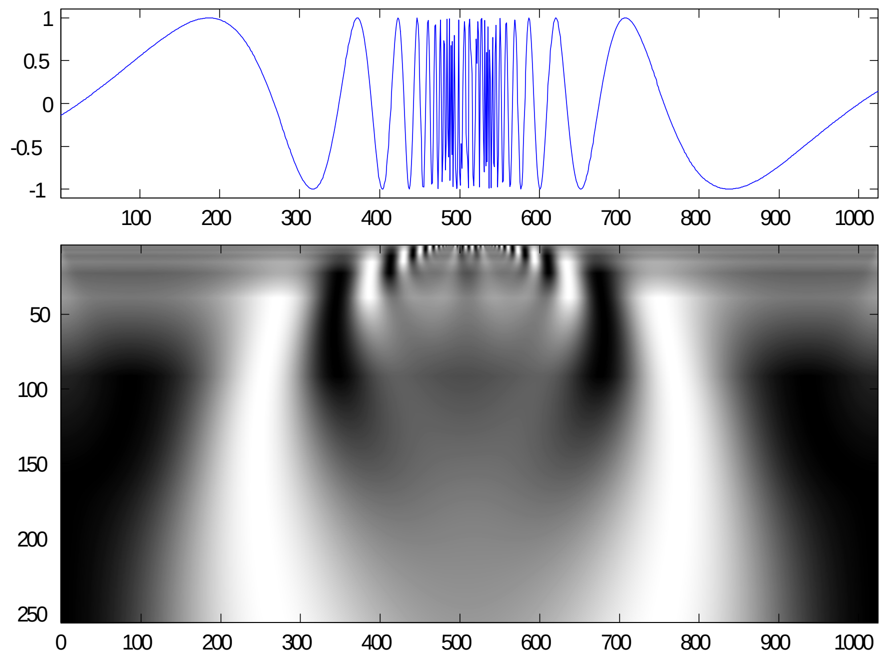
which represents a bounded signal that has an oscillating singularity at 0. Analysing with a Gabor wavelet, $\psi(t) = g(t)\, e^{2\pi i \eta t}$ we learned in the previous section that the modulus maxima of the normalised scalogram (i.e, $\frac{1}{\sqrt{a}} |W_\psi f(a,b)|$) appear at the ridge points $(a, b)$ where

$$a = \frac{\eta - \omega_0}{\phi'(b)}$$

where $\omega_0$ is the point where $\widehat{g}(\omega)$ assumes its maximum. For $f(t) = \sin\frac{1}{t}$, $\phi(t) = \frac{1}{t} + \frac{\pi}{2}$ and $\phi'(b) = -1/b^2$. The ridge points are located on the parabola

$$a = (\omega_0 - \eta)\, b^2.$$

Of course, points on this parabola are outside the cone of influence at 0. The following figures shows that high amplitude coefficients are indeed produced on this parabola.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