III. The Discrete Wavelet Transform

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1 Introduction

Developing the discrete wavelet transform involves the transition from the continuous time domain to the discrete time domain. Signals are replaced by sequences which may be thought of as samples of the signal at discrete time steps. The more accurate point of view, however, is not to consider these sequences as signal samples but as coefficients of approximations of the signal in a suitable space V_J of a multiresolution analsysis. This means that care should be taken when collecting signal samples. We will not dwell on this point anyfurther. In the discrete time domain, the scaling filter h and the wavelet filter g naturally replace the scaling function φ and the wavelet ψ.

In this chapter we detail the transition from continuous time to discrete time.

2 Notation and terminology

We begin by introducing some definitions and notations. We will use lower case letters c, d, ... etc to denote infinite sequences of complex numbers of the form

$$\begin{aligned} c &= (c_n) = (\ldots, c_{-2}, c_{-1}, c_0, c_1, c_2, \ldots), \\ d &= (d_n) = (\ldots, d_{-2}, d_{-1}, d_0, d_1, d_2, \ldots), \ldots. \end{aligned}$$

The operation of addition of two sequences is defined by adding the corresponding componenets:

$$\begin{aligned} c + d &:= (c_n + d_n) \\ &= (\ldots, c_{-2} + d_{-2}, c_{-1} + d_{-1}, c_0 + d_0, c_1 + d_1, c_2 + d_2, \ldots). \end{aligned}$$

Similarly, the operation of multiplying a sequence by a number is defined by multiplication componentwise:

$$\begin{aligned} \lambda c &:= (\lambda c_n) \\ &= (\ldots, \lambda c_{-2}, \lambda c_{-1}, \lambda c_0, \lambda c_1, \lambda c_2, \ldots). \end{aligned}$$

Given a sequence $c = (c_n)$ we use the notation

$$\|c\|^2 := \sum_n |c_n|^2 \tag{1}$$

if the inifinite sum exists. If it does, we say that the sequence c is square summable. The space of all square summable sequences is denoted by $\ell^2(\mathbb{Z})$. With the two operations defined above, $\ell^2(\mathbb{Z})$ is a linear space. With the norm defined by (1), $\ell^2(\mathbb{Z})$ is a Hilbert space. The inner product associated with this norm is

$$\langle c, d\rangle = \langle (c_n), (d_n)\rangle = \sum_n c_n \overline{d}_n.$$

The space $\ell^2(\mathbb{Z})$ has a special set of orthonormal basis $\{\epsilon_k\}_{k\in\mathbb{Z}}$ where

$$\epsilon_k = \underbrace{(\ldots, 0, 1, 0, \ldots)}_{\substack{\uparrow \\ k^{\text{th}} \text{ position}}}.$$

This basis is called the standard basis. One can write any sequence $c \in \ell^2(\mathbb{Z})$ as

$$c = \sum_k c_k \epsilon_k.$$

The reflection and translation operators on $\ell^2(\mathbb{Z})$ take the forms

$$\begin{aligned} \rho c &= (c_{-n}), \\ \tau_k c &= (c_{n-k}), \end{aligned}$$

respectively. The dilation operator is given by

$$\delta_j c = (c_{2^j n}),$$

where it is understood that $c_{2^j n} = 0$ if $2^j n$ is not an integer. In particular,

$$\delta_1 c = (\ldots, c_{-4}, c_{-2}, c_0, c_2, c_4, \ldots),$$

while

$$\delta_{-1} c = (\ldots, c_{-2}, 0, c_{-1}, 0, c_0, 0, c_1, 0, c_2, \ldots).$$

Therefore, $\delta_1 c$ "throws away" all the odd indexed components of c while $\delta_{-1} c$ pads the sequence c with zeros. The operator δ_1 is called a downsampling operator and δ_{-1} is called an upsampling operator. The operators δ_j and δ_{-j} are no longer unitary (they are not even invertible), however, they are mutually adjoint:

$$\langle \delta_j c, d\rangle = \langle c, \delta_{-j} d\rangle.$$

The convolution of two sequences $c, d \in \ell^2(\mathbb{Z})$ is defined by

$$c * d = \left(\sum_k c_k \, d_{n-k}\right).$$

The space $L^2(0,1)$ will also be useful to us in what follows. There are two Fourier transforms associated with $L^2(0,1)$ and $\ell^2(\mathbb{Z})$. The discrete Fourier transform

$$\mathcal{F}_d : L^2(0,1) \to \ell^2(\mathbb{Z})$$

defined by

$$\mathcal{F}_d f = \widehat{f} = (\langle f, e_n \rangle)$$

and the Fourier series transform

$$\mathcal{F}_s : \ell^2(\mathbb{Z}) \to L^2(0,1)$$

defined by

$$\mathcal{F}_s c = \widehat{c} = \sum_n c_n e_n.$$

It is straight forward to show that $\mathcal{F}_s^* = \mathcal{F}_d = \mathcal{F}_s^{-1}$ and $\mathcal{F}_d^* = \mathcal{F}_s = \mathcal{F}_d^{-1}$. The familiar change of convlution to multiplication is still valid for these two Fourier transforms:

$$\begin{aligned} \widehat{c * d} &= \widehat{c}\widehat{d}, \\ \widehat{f * g} &= \widehat{f} \circledast \widehat{g}, \end{aligned}$$

where the notation in the last equation needs the following explanation:

$$f * g = \int_0^1 f(\tau)\, g(t-\tau)\, d\tau$$

with g (and f) extended outside the interval $(0,1)$ by periodicity, and $\widehat{f} \circledast \widehat{g}$ is the sequence obtained from $\widehat{f}$ and $\widehat{g}$ by componentwise multiplication.

The down sampling operator introduces aliasing as we shall see from the Fourier transforms.

Lemma 1

$$\widehat{\delta_1 c} = \sqrt{2} D_{-1} \left(\widehat{c} + T_{1/2} \widehat{c} \right), \tag{2}$$

$$\widehat{\delta_{-1} c} = \frac{1}{\sqrt{2}} D_1 \widehat{c}. \tag{3}$$

Proof. (of (2)).

$$\begin{aligned} \widehat{c} + T_{1/2}\widehat{c} &= \sum_n c_n e_n + \sum_n (-1)^n c_n e_n \\ &= 2 \sum c_{2n} e_{2n} \\ &= \sqrt{2} D_1 \sum c_{2n} e_n \\ &= \sqrt{2} D_1 \widehat{\delta_1 c}. \end{aligned}$$

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3 The Approximation and Detail Operators and their adjoints

Let $\{V_j\}_{j\in\mathbb{Z}}$ be an MRA with scaling function φ, mother wavelet ψ, scaling filter $h = (h_n)$ and wavelet filter $g = (g_n)$. Recall that $h, g \in \ell^2(\mathbb{Z})$. In order to work efficiently with sequences, we need to introduce the operators $A_j,\ B_j : L^2(\mathbb{R}) \to \ell^2(\mathbb{Z})$ associated with the MRA which are difined by

$$\begin{aligned} A_j f &= \left(\langle f, \varphi_{j,n}\rangle\right), \\ B_j f &= \left(\langle f, \psi_{j,n}\rangle\right)\ \forall f \in L^2(\mathbb{R}). \end{aligned}$$

So for each function $f \in L^2(\mathbb{R})$, $A_j f$ gives the sequence of coefficients of $P_j f$ whereas $B_j f$ gives the sequence of coefficients of $Q_j f$.

Exercise 1 Obtain fromulas for A_j^* and B_j^* and show that

(a) for all $f \in L^2(\mathbb{R})$,

$$\begin{aligned} A_j^* A_j f &= P_j f, \\ B_j^* B_j f &= Q_j f; \end{aligned}$$

(b) for all sequences c in the range of A_j (range of B_j),

$$\begin{aligned} A_j A_j^* c &= c, \\ B_j B_j^* c &= c \end{aligned}$$

Define the convolution operators $H, G : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ by

$$\begin{aligned} Hc &= (\langle c, \tau_n h\rangle) = c * \rho\overline{h}, \\ Gc &= (\langle c, \tau_n g\rangle) = c * \rho\overline{g}. \end{aligned}$$

Exercise 2 Show that the adjoints of H, G are given by

$$H^* c = c * h,\ G^* c = c * g,$$

respectively.

Observe the following Fourier transforms:

$$\begin{aligned} m_0(\omega) &= \mathcal{F}_s h = \widehat{h} = \sum_n h_n e_n, \\ m_1(\omega) &= \mathcal{F}_s g = \widehat{g} = \sum_n g_n e_n. \end{aligned}$$

Exercise 3 Show that $T_n D_j = D_j T_{2^j n}$.

Lemma 2 *(two scale identities)*

(i) $\varphi_{j,n} = \sum_k h_{k-2n} \varphi_{j+1,k}$

(ii) $\psi_{j,,n} = \sum_k g_{k-2n} \varphi_{j+1,k.}$

Proof. It suffices to show (i) only. Since $\varphi_{j,n} \in V_{j+1}$,

$$\begin{aligned}
\varphi_{n,j} &= \sum_k \left\langle \varphi_{j,n}, \varphi_{j+1,k} \right\rangle \varphi_{j+1,k} = D_j T_n h_k D_1 T_k \varphi \\
&= \sum_k \langle D_j T_n \varphi, D_{j+1} T_k \varphi \rangle \varphi_{j+1,k} = \sum_k \langle T_n \varphi, D_1 T_k \varphi \rangle \varphi_{j+1,k} \\
&= \sum_k \langle \varphi, T_{-n} D_1 T_k \varphi \rangle \varphi_{j+1,k} = \sum_k \langle \varphi, D_1 T_{k-2n} \varphi \rangle \varphi_{j+1,k} \\
&= \sum_k \left\langle \varphi, \varphi_{1,k-2n} \right\rangle \varphi_{j+1,k} = \sum_k h_{k-2n} \varphi_{j+1,k}.
\end{aligned}$$

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We then have the following recursions.

Lemma 3 *(decomposition recursion relations)*
The following decomposition relations hold:

$$\begin{aligned}
A_j f &= \delta_1 H A_{j+1} f, \qquad (4) \\
B_j f &= \delta_1 G A_{j+1} f \qquad (5)
\end{aligned}$$

for all $f \in L^2(\mathbb{R})$.

Proof. We show only (4). For any $n \in \mathbb{Z}$,

$$\begin{aligned}
A_j f &= \left(\left\langle f, \varphi_{n,j} \right\rangle \right) = \left(\left\langle f, \sum_k h_{k-2n} \varphi_{k,j+1} \right\rangle \right) \\
&= \left(\sum_k \overline{h}_{k-2n} \left\langle f, \varphi_{k,j+1} \right\rangle \right) = \left(\langle A_{j+1} f, \tau_{2n} h \rangle \right) \\
&= \delta_1 \left(\langle A_{j+1} f, \tau_n h \rangle \right) \\
&= \delta_1 H A_{j+1} f.
\end{aligned}$$

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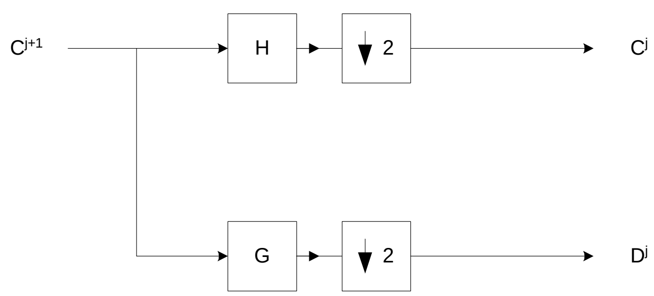
The decomposition process is often depicted as shown below.



We should remark here that equations (4, 5) express the fact that the approximation and detail coefficients at the coarser resolution j can be directly computed from the approximation coefficients at the finer resolution $j+1$ without having to recalculate the discretization coefficients. In other words, $P_j f$ and $Q_j f$ can be computed directly from $P_{j+1} f$. We will proceed to show that this process is completely reversible: the approximation coefficients at the finer resolution $j+1$ can be directly computed from the approximation and detail coefficients at the next coarser resolution j.

Lemma 4 *(reconstruction recursion relation)*
At the reconstruction level

$$A_{j+1} f = H^* \delta_{-1} A_j f + G^* \delta_{-1} B_j f. \tag{6}$$

Proof. Since

$$\begin{aligned} P_{j+1} &= P_j + Q_j, \\ A_{j+1}^* A_{j+1} &= A_j^* A_j + B_j^* B_j, \end{aligned}$$

using Lemma 3,

$$\begin{aligned} A_j^* &= (\delta_1 H A_{j+1})^* \\ &= A_{j+1}^* H^* \delta_{-1} \end{aligned}$$

and

$$B_j^* = A_{j+1}^* G^* \delta_{-1}.$$

Hence,

$$A_{j+1}^* A_{j+1} = A_{j+1}^* \left(H^* \delta_{-1} A_j + G^* \delta_{-1} B_j \right).$$

Multiplying both sides of the above equation by A_{j+1} (and using Exercise 2) gives,

$$A_{j+1} = H^* \delta_{-1} A_j + G^* \delta_{-1} B_j. \tag{7}$$

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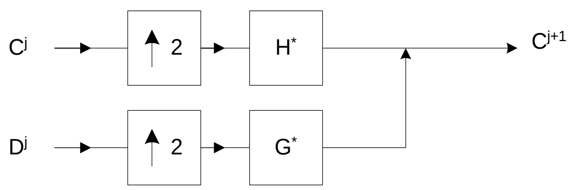
The reconstruction process is often depicted as follows:



Equation (7) expresses the perfect reconstruction of a signal at a higher resolution level (finer scale) from its approximation and detail at a lower resolution (coarser scale) level.

Operations count: Given an approximation $A_{j+1} f$ of a signal of length N, the decomposition step into approximation and detail (equations 4, 5) costs about $2N$ multiplications and additions because we do not need to compute the discarded terms when downsampling. After the decomposition step, $A_j f$ and $B_j f$ both have length $N/2$. Therefore, the reconstruction step (7) costs about $4N/2 = 2N$ multiplications and additions. The operation count for the discrete wavelet algorithm is, thus, $O(N)$ which is a substantial improvement over the Fast Fourier Transform in which the operation count is $O(N \log N)$.

4 Further Properties of the QMF

The Fourier transforms involving these operators are given below.

Lemma 5 *(Fourier transform properties)*
The following Fourer transforms hold:

$$\begin{aligned}
\widehat{\delta_1 H c} &= \sqrt{2} D_{-1}\left(\widehat{c}\,\overline{m}_0 + T_{1/2}\left(\widehat{c}\,\overline{m}_0\right)\right), \\
\widehat{\delta_1 G c} &= \sqrt{2} D_{-1}\left(\widehat{c}\,\overline{m}_1 + T_{1/2}\left(\widehat{c}\,\overline{m}_1\right)\right), \\
\widehat{H^*\delta_{-1} c} &= \frac{1}{\sqrt{2}}\left(D_1 \widehat{c}\right) m_0, \\
\widehat{G^*\delta_{-1} c} &= \frac{1}{\sqrt{2}}\left(D_1 \widehat{c}\right) m_1.
\end{aligned}$$

Proof. We will show the first of these identities.

$$\begin{aligned}
\widehat{\delta_1 H c} &= \sqrt{2} D_{-1}\left(\widehat{Hc} + T_{1/2}\widehat{Hc}\right) = \sqrt{2} D_{-1}\left(\widehat{c * \rho\overline{h}} + T_{1/2}\widehat{c * \rho\overline{h}}\right) \\
&= \sqrt{2} D_{-1}\left(\widehat{c}\,\overline{m}_0 + T_{1/2}\left(\widehat{c}\,\overline{m}_0\right)\right).
\end{aligned}$$

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Exercise 4 Work out the details of the remaining three relations in the previous lemma.

Next we investigate the relationships between the operators defined above and the identities involving the functions H and G.

Theorem 6 *(correspondence between auxiliary functions and operators)*

(i) *The conditions* $|m_0|^2 + T_{1/2}|m_0|^2 = |m_0|^2 + T_{1/2}|m_1|^2 = 1$ *are equivalent to*

$$\delta_1 H H^* \delta_{-1} = \delta_1 G G^* \delta_{-1} = I.$$

(ii) *The condition* $m_1\overline{m}_0 + T_{1/2} m_1 \overline{m}_0 = 0$ *is equivalent to*

$$\delta_1 G H^* \delta_{-1} = \delta_1 H G^* \delta_{-1} = 0.$$

(iii) *The conditions* $|m_0|^2 + T_{1/2}|m_0|^2 = 1$, $m_1 = Q\overline{m}_0$ *with* $|Q| = 1$ *and* $Q + T_{1/2}Q = 1$ *are equivalent to*

$$\delta_{-1}\left(H^*H + G^*G\right)\delta_1 = I.$$

Proof. To prove the first equivalence in (i), we calsulate

$$\begin{aligned}
\delta_1 \widehat{H H^* \delta_{-1}} c &= \sqrt{2} D_{-1}\left(\widehat{H^*\delta_{-1}} c\,\overline{m}_0 + T_{1/2}\left(\widehat{H^*\delta_{-1}} c\,\overline{m}_0\right)\right) \\
&= D_{-1}\left(\left(D_1\widehat{c}\right) m_0\,\overline{m}_0 + T_{1/2}\left(\left(D_1\widehat{c}\right)\ m_0\,\overline{m}_0\right)\right) \\
&= D_{-1}\left(D_1\widehat{c}\ |m_0|^2 + T_{1/2}\left(D_1\widehat{c}\right)\ |m_0|^2\right) \\
&= D_{-1}\left(D_1\widehat{c}|m_0|^2 + T_{1/2}\left(D_1\widehat{c}\right) T_{1/2}|m_0|^2\right) \\
&= D_{-1}\left(D_1\widehat{c}\ |m_0|^2 + \left(D_1 T_1 \widehat{c}\right) T_{1/2}|m_0|^2\right) \\
&= \widehat{c}\left(|m_0|^2 + T_{1/2}|m_0|^2\right).
\end{aligned}$$

If $|m_0|^2 + T_{1/2}|m_0|^2 = 1$, then $\delta_1 \widehat{HH^*\delta_{-1}c} = \widehat{c}$, which means that $\delta_1 HH^*\delta_{-1} = I$. On the other hand, if $\delta_1 HH^*\delta_{-1} = I$, then $\delta_1 \widehat{HH^*\delta_{-1}c} = \widehat{c}$ for all $c \in \ell^2(\mathbb{Z})$. In particular, for $c = \epsilon_0$ $(\widehat{c} = 1)$,

$$|m_0|^2 + T_{1/2}|m_0|^2 = 1.$$

To prove the equivalence in (iii) we proceed with similar calculation to that of part (i):

$$\begin{aligned}
\delta_{-1}(H^*\widehat{H + G^*G})\delta_1 c &= \widehat{c}\,|m_0|^2 + m_0 T_{1/2}(\widehat{c}\,\overline{m}_0) + \widehat{c}|m_1|^2 + m_1 T_{1/2}\widehat{c}\,\overline{m}_1 \\
&= \left(\widehat{c}\left(|m_0|^2 + |m_1|^2\right) + T_{1/2}\widehat{c}\left(m_0 T_{1/2}\overline{m}_0 + m_1 T_{1/2}\overline{m}_1\right)\right).
\end{aligned}$$

If $|m_0|^2 + T_{1/2}|m_0|^2 = 1$ and $m_1 = QT_{1/2}\overline{m}_0$ with $|Q| = 1$, $T_{1/2}Q = -Q$ then

$$\begin{aligned}
m_0 T_{1/2}\overline{m}_0 + m_1 T_{1/2}\overline{m}_1 &= m_0 T_{1/2}\overline{m}_0 + QT_{1/2}\overline{m}_0 T_{1/2}\left(\overline{Q}T_{1/2}m_0\right) \\
&= m_0 T_{1/2}\overline{m}_0 + Qm_0 T_{1/2}\overline{m}_0 T_{1/2}\overline{Q} \\
&= m_0 T_{1/2}\overline{m}_0\left(1 + QT_{1/2}\overline{Q}\right) \\
&= m_0 T_{1/2}\overline{m}_0\left(1 - |Q|^2\right) = 0
\end{aligned}$$

and

$$|m_0|^2 + |m_1|^2 = |m_0|^2 + T_{1/2}|m_0|^2 = 1.$$

Therefore,

$$\delta_{-1}(H^*\widehat{H + G^*G})\delta_1 c = \widehat{c}$$

from which we get

$$\delta_{-1}(H^*H + G^*G)\delta_1 = I.$$

On the other hand, if $\delta_{-1}(H^*H + G^*G)\delta_1 = I$ then

$$\widehat{c} = \left(\widehat{c}\left(|m_0|^2 + |m_1|^2\right) + T_{1/2}\widehat{c}\left(HT_{1/2}\overline{m}_0 + m_1 T_{1/2}\overline{m}_1\right)\right)$$

for all $c \in \ell^2(\mathbb{Z})$. Choosing $c = \epsilon_0$ and $c = \epsilon_1$, we obtain

$$\begin{aligned}
1 &= |m_0|^2 + |m_1|^2 + T_{1/2}\left(m_0 T_{1/2}\overline{m}_0 + m_1 T_{1/2}\overline{m}_1\right), \\
e_1 &= e_1\left(|m_0|^2 + |m_1|^2\right) + T_{1/2}\left(e_1\left(m_0 T_{1/2}\overline{m}_0 + m_1 T_{1/2}\overline{m}_1\right)\right) \\
&= e_1\left(|m_0|^2 + |m_1|^2 - T_{1/2}\left(m_0 T_{1/2}\overline{m}_0 + m_1 T_{1/2}\overline{m}_1\right)\right).
\end{aligned}$$

Solving this system we get

$$|m_0|^2 + |m_1|^2 = 1, \qquad (8)$$

$$m_0 T_{1/2}\overline{m}_0 + m_1 T_{1/2}\overline{m}_1 = 0. \qquad (9)$$

The second of these equations gives

$$m_0 T_{1/2}\overline{m}_0 = -m_1 T_{1/2}\overline{m}_1.$$

Taking the square moduli on both sides, substituting from equation (8) and simplifying we get

$$|m_1|^2 = T_{1/2}|m_0|^2.$$

Then

$$m_1 = QT_{1/2}\overline{m}_0,$$

where $|Q| = 1$ and $T_{1/2}Q = -Q$. ■

We remark that the aliasing introduced by the downsampling operator is completely negated at the reconstruction step because of the property

$$m_0 T_{1/2}\overline{m}_0 + m_1 T_{1/2}\overline{m}_1 = 0.$$

Exercise 5 Prove part (ii) of the previous theorem.

Other properties of H and G in terms of the sequences h and g are given in what follows.

1. Integrating the equation
$$|m_0|^2 + T_{1/2}|m_0|^2 = 1 \tag{10}$$
between 0 and 1 gives
$$\|m_0\|^2_{L^2(0,1)} = \frac{1}{2}.$$
Therefore,
$$\|h\|^2 = 1,$$
or
$$\sum |h_n|^2 = 1.$$
Equivalently,
$$\|\varphi\| = 1.$$
Since we also have a similar equation for m_1,
$$|m_1|^2 + T_{1/2}|m_1|^2 = 1,$$
we similarly conclude that
$$\begin{aligned}\|m_1\|^2_{L^2(0,1)} &= \frac{1}{2},\\ \|g\|^2 &= 1,\\ \sum |g_n|^2 &= 1,\\ \|\psi\| &= 1.\end{aligned}$$

2. Taking the discrete Fourier transform of (10) gives
$$\left(h * \rho\overline{h}\right)_n + (-1)^n \left(h * \rho\overline{h}\right)_n = \delta_{n0}.$$
Therefore,
$$\left(h * \rho\overline{h}\right)_{2n} = \frac{1}{2}\delta_{n0}.$$
Explicitly,
$$\sum h_k \overline{h}_{k-2n} = \delta_{n0}.$$

3. The equation $m_0(0) = 1$ gives $\sum h_n = \sqrt{2}$.
4. The equation $m_1(0) = 0$ gives $\sum g_n = 0$.
5. The equation $m_0\left(\frac{1}{2}\right) = 0$ gives $\sum (-1)^n h_n = 0$, which is equivalent to $\sum h_{2n} = \sum h_{2n+1}$.
6. Taking the discrete Fourier transform of the equation

$$m_1 \overline{m}_0 + T_{1/2} m_1 \overline{m}_0 = 0$$

   yields

$$\left(g * \rho\overline{h}\right)_n + (-1)^n \left(g * \rho\overline{h}\right)_n = 0.$$

   Therefore,

$$\left(g * \rho\overline{h}\right)_{2n} = 0.$$

   Explicitly

$$\sum_k g_k \overline{h}_{k-2n} = 0.$$

7. Taking the discrete Fourier transform of the equation

$$|m_0|^2 + |m_1|^2 = 1$$

   yields

$$h * \rho\overline{h} + g * \rho\overline{g} = \delta_0.$$

   Since

$$\begin{aligned} \left(h * \rho\overline{h}\right)_{2n} &= \delta_{n0}, \\ (g * \rho\overline{g})_{2n} &= \delta_{n0}. \end{aligned}$$

   Explicitly,

$$\sum_k g_k \overline{g}_{k-2n} = \delta_{n0}.$$