III The Discrete Wavelet Transform

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1 Wavelet Bases

The continuous wavelet transform offers the capability of analysing the local behvior of a signal. The translations and dilations of wavelets

$$D_a T_b \psi$$

for $a > 0$ and $b \in \mathbb{R}$ provide "more than enough" bases through which reconstruction is possible. The purpose of this chapter is to develop a set of bases consisting of wavelets which will span $L^2(\mathbb{R})$ and at the same time retain the capability of local signal analysis. This will be accomplished by restricting the scales a to the set $\{2^j\}_{j\in\mathbb{Z}}$ and the translations b to the set $\{k\}_{k\in\mathbb{Z}}$. Thus $\psi_{j,k}$ will denote

$$D_j T_k \psi(t) = \sqrt{2^j}\,\psi\left(2^j (t-k)\right).$$

Observe the slightly changed definition of the dilation operator. Now higher values of j stand for higer frequencies, or small scales.

2 Multiresolution Analysis

Roughly speaking, a multiresolution analysis is the representation of a signal f by a sequence of signals which capture progressively finer details of f. To introduce the exact definition of a multiresolution analysis we need first the follwoing basic concepts and notation.

The frequency modulators e_n For convenience we will define the frequency modulator functions e_n by

$$e_n(\omega) = e^{2\pi i n\omega}, \ \forall \omega \in \mathbb{R}.$$

Dense Subspace A subspace M of $L^2(\mathbb{R})$ is said to be dense in $L^2(\mathbb{R})$ if given any $f \in L^2(\mathbb{R})$ and any $\epsilon > 0$ there exists a $g \in M$ such that

$$\|f - g\| < \epsilon.$$

Closed Subspace A subspace M of $L^2(\mathbb{R})$ is said to be closed in $L^2(\mathbb{R})$ if given $f \in L^2(\mathbb{R})$ there exists a (necessarily unique) function $g \in M$ such that

$$\langle f - g, h\rangle = 0 \ \forall h \in M. \tag{1}$$

Orthogonal projections Let M be a closed subspace of $L^2(\mathbb{R})$. The orthogonal projection $P_M : L^2(\mathbb{R}) \to M$ is defined by $P_M f = g$, where g is the function in (1). The most important properties of P_M are:

(i) $P_M^2 = P_M$.

(ii) $\langle P_M f, g\rangle = \langle f, P_M g\rangle \ \forall f, g \in L^2(\mathbb{R})$.

The Span of a set of functions The span of a given sequence of functions $\{f_n\}_{n\in\mathbb{Z}}$, say in $L^2(\mathbb{R})$, is defined to be the set of finite linear combinations of elements of $\{f_n\}_{n\in\mathbb{Z}}$. Put differently, we say that

$$N = \operatorname{span}\{f_n\}_{n\in\mathbb{Z}}$$

if every element $g \in N$ can be written in the form

$$g = \alpha_1 f_{n_1} + \alpha_2 f_{n_1} + \ldots + \alpha_m f_{n_m}$$

where m and the complex constants $\alpha_1, \alpha_2, \ldots, \alpha_m$ possibly depend on g. Observe that N is itself a subspace of $L^2(\mathbb{R})$.

Orthonormal basis (ONB) Let M be a (closed) subspace of $L^2(\mathbb{R})$. The orthonormal sequence of functions $\{\eta_n\}_{n\in\mathbb{Z}}$ is called an orthonormal basis for M if every $f \in M$ has the unique decomposition

$$f = \sum_n \langle f, \eta_n\rangle \eta_n.$$

We are now in a position to introduce the definition of a multiresolution analysis.

Definition 1 *(multiresolution analysis MRA)*
A sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ is called a multiresolution analysis if

1. $V_j \subseteq V_{j+1} \ \forall j \in \mathbb{Z}$.
2. $\cup_{j\in\mathbb{Z}} V_j$ *is dense in* $L^2(\mathbb{R})$.
3. $\cap_{j\in\mathbb{Z}} V_j = \{0\}$.
4. $f \in V_j$ *if and only if* $D_1 f \in V_{j+1}$.
5. *There exists a function* $\varphi \in V_0$ *called the associated scaling function such that* $\{T_n\varphi\}_{n\in\mathbb{Z}}$ *forms an ONB for* V_0.

The following projections are associated with an MRA.

Definition 2 *(approximation and detail operators)*
Suppose $\{V_j\}_{j\in\mathbb{Z}}$ is a multiresolution analysis.

(i) *The sequence of orthogonal projections $P_j := P_{V_j}$ for all $j \in \mathbb{Z}$ is called the sequence of approximation operators.*

(ii) *The sequence of orthogonal projections $Q_j := P_{j+1} - P_j$ for all $j \in \mathbb{Z}$ is called the sequence of detail operators.*

It follows from the definition of an MRA that for :

1. $\|P_j f\| \leq \|f\|$ for all $j \in \mathbb{Z}$ and all $f \in L^2(\mathbb{R})$.
2. $\|P_{j+1} f\| \geq \|P_j f\|$ for all $j \in \mathbb{Z}$ and all $f \in L^2(\mathbb{R})$.
3. For all $f \in L^2(\mathbb{R})$, $P_j f \to 0$ as $j \to -\infty$.
4. For all $f \in L^2(\mathbb{R})$, $P_j f \to f$ as $j \to \infty$.
5. For every $f \in V_0$ we can write

$$f = \sum_n \langle f, T_n \varphi \rangle T_n \varphi.$$

In an MRA, the sequence $\{T_n \varphi\}_{n \in \mathbb{Z}}$ is called an orthonormal system of translates. What conditions should a function $\varphi \in L^2(\mathbb{R})$ satisfy in order that $\{T_n \varphi\}_{n \in \mathbb{Z}}$ be an orthonormal system of translates? The following lemma gives the answer.

Lemma 3 *(conditions for orthonormal systems of translates)*
$\{T_n \varphi\}_{n \in \mathbb{Z}}$ *is an orthnormal system of translates iff*

$$\sum_n |\widehat{\varphi}(\omega + n)|^2 = 1 \ \forall \omega \in [-1, 1]. \tag{2}$$

Proof. Suppose $\{T_n \varphi\}_{n \in \mathbb{Z}}$ is an orthnormal system of translates. Then

$$\begin{aligned}
\delta_{k,0} &= \langle T_k \varphi, \varphi \rangle = \left\langle \widehat{T_k \varphi}, \widehat{\varphi} \right\rangle \\
&= \langle e_k \widehat{\varphi}, \widehat{\varphi} \rangle = \int_{-\infty}^{\infty} |\widehat{\varphi}(\omega)|^2 e^{2\pi i k \omega} d\omega \\
&= \sum_n \int_n^{n+1} |\widehat{\varphi}(\omega)|^2 e^{2\pi i k \omega} d\omega \\
&= \sum_n \int_0^1 |\widehat{\varphi}(\omega + n)|^2 e^{2\pi i k \omega} d\omega \\
&= \int_0^1 \sum_n |\widehat{\varphi}(\omega + n)|^2 e^{2\pi i k \omega} d\omega.
\end{aligned}$$

The sequence $\{\delta_{k,0}\}_{k \in \mathbb{Z}}$ is nothing but the Fourier series coefficients for the function 1. Therefore, by the uniqueness of the Fourier series, $\sum_n |\widehat{\varphi}(\omega + n)|^2 = 1 \ \forall \omega \in [0, 1]$. The same steps can be repeated to obtain a Fourier series expansion on $[-1, 0]$ and get $\sum_n |\widehat{\varphi}(\omega + n)|^2 = 1 \ \forall \omega \in [-1, 0]$.

The only if part can be shown by reversing the above steps. ■

In the case of compactly supported functions, condition (2) can be relaxed as follows.

Lemma 4 *(relaxed scaling functions)*

Suppose $\varphi_1 \in L^2(\mathbb{R})$ has compact support and satisfies

$$A \leq \sum_n |\widehat{\varphi}_1(\omega + n)|^2 \leq B \ \forall \omega \in [-1, 1]$$

and for some $A, B > 0$. Then there is a function $\varphi \in L^2(\mathbb{R})$ such that

(i) $\{T_n \varphi\}_{n \in \mathbb{Z}}$ *is an orthnormal system of translates.*

(ii) $\overline{\operatorname{span}}\{T_n \varphi\} = \overline{\operatorname{span}}\{T_n \varphi_1\}$.

The functions φ_1 and φ in the above lemma are related by

$$\widehat{\varphi}(\omega) = \frac{\widehat{\varphi}_1(\omega)}{\sqrt{\sum_n |\widehat{\varphi}_1(\omega + n)|^2}}.$$

Typically, an MRA is constructed by choosing a function $\varphi \in L^2(\mathbb{R})$ satisfying (2), defining the space V_0 by

$$V_0 = \overline{\operatorname{span}}\{T_n \varphi\}, \tag{3}$$

the spaces V_j by

$$V_j = \{D_j f : f \in V_0\}, \ j \in \mathbb{Z}, \tag{4}$$

and then proving that Conditions 1-3 of Definition 1 hold.

Exercise 1 Define φ_{jk} by $\varphi_{jk} = D_j T_k \varphi$. Prove that, for each $j \in \mathbb{Z}$, $\{\varphi_{jk}\}_{k \in \mathbb{Z}}$ is an orthonormal basis for V_j.

3 Properties of the Scaling Function

We present in this section some important properties of the scaling function for an MRA.

Theorem 5 *(necessary condition for the scaling function)*

Suppose $\{V_j\}_{j \in \mathbb{Z}}$ is an MRA in $L^2(\mathbb{R})$ with associated scaling function $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then $\widehat{\varphi}$ is continuous and

$$\left| \int_{-\infty}^{\infty} \varphi(t)\, dt \right| = 1$$

Proof. Let $f \in L^2(\mathbb{R})$ be such that $\widehat{f}$ is continuous and $\operatorname{supp} \widehat{f} \subset [-R, R]$. Then since $\{\varphi_{jk}\}_{k \in \mathbb{Z}}$ is an orthonormal basis for V_j,

$$\begin{aligned} P_j f &= \sum_k \langle f, \varphi_{jk} \rangle \varphi_{jk} \\ &= \sum_k \left\langle \widehat{f}, D_{-j} e_k \widehat{\varphi} \right\rangle \varphi_{jk}. \end{aligned}$$

Taking the Fourier transform on both sides we get

$$\widehat{P_j f} = D_{-j}\left(\widehat{\varphi}\sum_k \left\langle \overline{\widehat{\varphi}} D_j \widehat{f}, e_k \right\rangle e_k\right).$$

Since supp $D_j \widehat{f} \subset \left[-R/2^j, R/2^j\right]$, then, for sufficiently large j, $R/2^j \leq 1$. Hence, the series on the right is the Fourier series expansion of $\overline{\widehat{\varphi}}\ D_{2^j}\widehat{f}$in $L^2(0,1)$. Hnence,

$$\begin{aligned}\widehat{P_j f} &= D_{-j}\left(\left|\widehat{\varphi}\right|^2 D_j \widehat{f}\right) \\ &= \sqrt{2^j} D_{-j}\left|\widehat{\varphi}\right|^2 \widehat{f}.\end{aligned}$$

Taking the limit as $j \to \infty$, and noting that $P_j f \to f$, $\widehat{P_j f} \to \widehat{f}$ and $\sqrt{2^j} D_{-j}\left|\widehat{\varphi}\right|^2(\omega) = \left|\widehat{\varphi}\right|^2\left(\frac{\omega}{2^j}\right) \to \left|\widehat{\varphi}(0)\right|^2$ we get

$$\widehat{f}(\omega) = \left|\widehat{\varphi}(0)\right|^2 \widehat{f}(\omega).$$

Therefore,

$$\left|\widehat{\varphi}(0)\right| = 1, \tag{5}$$

which is the same as

$$\left|\int_{-\infty}^{\infty} \varphi(t)\, dt\right| = 1.$$

■

Corollary 6 *(properties of φ)*

Assume $\{V_j\}_{j\in\mathbb{Z}}$ is an MRA in $L^2(\mathbb{R})$ with associated scaling function φ. Then $\widehat{\varphi}(n) = 0$ for all $n \in \mathbb{Z}$.

Proof. This follows immediately from (2) and (5). ■

Lemma 7 *(the two scale dilation equation)*

Suppose $\{V_j\}_{j\in\mathbb{Z}}$ is an MRA in $L^2(\mathbb{R})$ with associated scaling function φ. Then there exists a square summable sequence $\{h_k\}_{k\in\mathbb{Z}}$ such that

$$\varphi(t) = \sum_k h_k \varphi_{1,k}(t). \tag{6}$$

Furthermore,

$$\widehat{\varphi}(\omega) = m_0\left(\frac{\omega}{2}\right)\widehat{\varphi}\left(\frac{\omega}{2}\right), \tag{7}$$

where

$$m_0 = \sqrt{2}\sum_k h_k e_k \tag{8}$$

and where the infinite sum exists for all $\omega \in \mathbb{R}$.

Proof. Equation (6) follows immediately from the facts that $V_0 \subset V_1$ and that $\{\varphi_{1,k}\}_{k\in\mathbb{Z}}$ is an orthomormal basis for V_1. We explicitly have

$$h_k = \langle \varphi, \varphi_{1,k} \rangle \ \forall k \in \mathbb{Z}.$$

Taking the Fourier transform on both sides of (6) gives

$$\begin{aligned}
\widehat{\varphi}(\omega) &= \sum_k h_k \widehat{D_1 T_k \varphi}(\omega) \\
&= \sum_k h_k D_{-1} \widehat{T_k \varphi}(\omega) \\
&= \sum_k h_k D_{-1} e^{2\pi i k \omega} \widehat{\varphi}(\omega) \\
&= \sqrt{2} \sum_k h_k e^{\pi i k \omega} \widehat{\varphi}\left(\frac{\omega}{2}\right) \\
&= m_0\left(\frac{\omega}{2}\right) \widehat{\varphi}\left(\frac{\omega}{2}\right).
\end{aligned}$$

This is equation (7) with the function m_0 given by (8). ■

Some elementary properties of the function m_0 are given in the following lemma.

Lemma 8 *(properties of m_0)*

Let m_0 be the function defined by (8). Then the following properties hold.

(i) m_0 *is a periodic function of period* 1.

(ii) $m_0(n) = 1 \ \forall n \in \mathbb{Z}$.

(iii) *We have*

$$|m_0(\omega)|^2 + |T_{1/2} m_0(\omega)|^2 = 1 \ \forall \omega \in \mathbb{R}.$$

(iv) $m_0\left(\frac{2n+1}{2}\right) = 0 \ \forall n \in \mathbb{Z}$.

Proof. We only show (iii). Since $\{T_n \varphi\}$ is an orthonormal system of translates, $\forall \omega \in [-1, 1]$

$$\begin{aligned}
1 &= \sum_n |\widehat{\varphi}(\omega + n)|^2 = \sum_n \left| m_0\left(\frac{\omega+n}{2}\right)\right|^2 \left|\widehat{\varphi}\left(\frac{\omega+n}{2}\right)\right|^2 \\
&= \sum_k \left| m_0\left(\frac{\omega+2k}{2}\right)\right|^2 \left|\widehat{\varphi}\left(\frac{\omega+2k}{2}\right)\right|^2 + \sum_k \left| m_0\left(\frac{\omega+2k+1}{2}\right)\right|^2 \left|\widehat{\varphi}\left(\frac{\omega+2k+1}{2}\right)\right|^2 \\
&= \left| m_0\left(\frac{\omega}{2}\right)\right|^2 \sum_k \left|\widehat{\varphi}\left(\frac{\omega}{2} + k\right)\right|^2 + \left| m_0\left(\frac{\omega+1}{2}\right)\right|^2 \sum_k \left|\widehat{\varphi}\left(\frac{\omega+1}{2} + k\right)\right|^2 \\
&= \left| m_0\left(\frac{\omega}{2}\right)\right|^2 + \left| m_0\left(\frac{\omega+1}{2}\right)\right|^2.
\end{aligned}$$

For $\omega \in [-1/2, 1/2]$ we may replace ω by 2ω in the above equation, which yields the result for $\omega \in [-1/2, 1/2]$. The period 1 of m_0 implies the result for all $\omega \in \mathbb{R}$. ■

Definition 9 *(scaling filter)*

Suppose $\{V_j\}_{j\in\mathbb{Z}}$ *is an MRA in* $L^2(\mathbb{R})$ *with associated scaling function* φ*. The sequence* $\{h_k\}_{k\in\mathbb{Z}}$ *is called the scaling filter associated with* φ*. The function* $m_0(\omega)$ *given by (8) is called the auxiliary function associated with* φ*.*

3.1 Orthomormal Wavelet Bases

We show in this section how an MRA gives rise to a wavelet analysis. So, we assume that $\{V_j\}_{j\in\mathbb{Z}}$ is an MRA in $L^2(\mathbb{R})$ with associated scaling function φ. Since $V_j \subset V_{j+1}$ we may write

$$\begin{aligned} V_{j+1} &= V_j \oplus W_j \\ &= V_{j-1} \oplus W_{j-1} \oplus W_j \\ &= \cdots \\ &= \bigoplus_{k=-\infty}^{j} W_k \end{aligned}$$

and thus, we have the decompositoin of $L^2(\mathbb{R})$ into a seuquence of orthogonal subspaces

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j.$$

We want to show that the sequence of subspaces $\{W_j\}_{j\in\mathbb{Z}}$ is spanned by the dilations and translations of a single function ψ. This function is called

Exercise 2 Prove that if $\{T_n\psi\}_{n\in\mathbb{Z}}$ is an ONB for W_0 then $\{\psi_{ij}\}_{n\in\mathbb{Z}}$ is an ONB for W_j, where $\psi_{ij} = D_j T_n \psi$.

Thus it is enogh to construct the function ψ whose translations form an ONB for W_0. Since

$$W_0 = V_1 \ominus V_0,$$

ψ is required to satisfy the two conditions:

1. $\psi \in V_1$,
2. $\psi \perp V_0$.

The first of these two conditions implies that ψ can be written in terms of the basis $\{\varphi_{1,n}\}_{n\in\mathbb{Z}}$ for V_1, that is

$$\psi = \sum_n g_n D_1 T_n \varphi,$$

or,

$$D_{-1}\psi = \sum_n g_n T_n \varphi.$$

Taking the Fourier Transform on both sides we get

$$\begin{aligned} D_1\widehat{\psi} &= \sum_n g_n e_n \widehat{\varphi} \\ &= m_1 \widehat{\varphi}, \end{aligned}$$

where

$$m_1(\omega) = \sum_n g_n e^{2\pi i n \omega}. \tag{9}$$

Therefore,

$$\widehat{\psi} = D_{-1}(m_1 \widehat{\varphi}). \tag{10}$$

The second condition means

$$\langle \psi, T_n \varphi \rangle = 0 \ \forall n \in \mathbb{Z}.$$

From this condition we get the following theorem.

Theorem 10 *(determination of the function m_1)*
If ψ is orthogonal to V_0 then

$$m_1 \overline{m}_0 + T_{1/2}(m_1 \overline{m}_0) = 0 \tag{11}$$

on $\mathbb{R}$.

Proof. For any $n \in \mathbb{Z}$ we have

$$\begin{aligned} 0 &= \langle \psi, T_n \varphi \rangle = \left\langle \widehat{\psi}, e_n \widehat{\varphi} \right\rangle \\ &= \int_{-\infty}^{\infty} e^{-2\pi i n \omega} \widehat{\psi}(\omega) \overline{\widehat{\varphi}}(\omega) \, d\omega \\ &= \int_{-\infty}^{\infty} e^{-2\pi i n \omega} m_1\left(\frac{\omega}{2}\right) \widehat{\varphi}\left(\frac{\omega}{2}\right) \overline{\widehat{\varphi}}(\omega) \, d\omega \\ &= \int_{-\infty}^{\infty} e^{-2\pi i n \omega} m_1\left(\frac{\omega}{2}\right) \widehat{\varphi}\left(\frac{\omega}{2}\right) \overline{m}_0\left(\frac{\omega}{2}\right) \overline{\widehat{\varphi}}\left(\frac{\omega}{2}\right) d\omega \\ &= \int_{-\infty}^{\infty} e^{-2\pi i n \omega} m_1\left(\frac{\omega}{2}\right) \overline{m}_0\left(\frac{\omega}{2}\right) \left|\widehat{\varphi}\left(\frac{\omega}{2}\right)\right|^2 d\omega \\ &= \int_0^1 e^{-2\pi i n \omega} \sum_k m_1\left(\frac{\omega+k}{2}\right) \overline{m}_0\left(\frac{\omega+k}{2}\right) \left|\widehat{\varphi}\left(\frac{\omega+k}{2}\right)\right|^2 d\omega \\ &= \int_0^1 e^{-2\pi i n \omega} \left(m_1\left(\frac{\omega}{2}\right) \overline{m}_0\left(\frac{\omega}{2}\right) + m_1\left(\frac{\omega+1}{2}\right) \overline{m}_0\left(\frac{\omega+1}{2}\right) \right) d\omega. \end{aligned}$$

The last expression gives the Fourier coefficients for the function $m_1\left(\frac{\omega}{2}\right) \overline{m}_0\left(\frac{\omega}{2}\right) + m_1\left(\frac{\omega+1}{2}\right) \overline{m}_0\left(\frac{\omega+1}{2}\right)$. Since all these coefficients are zeros, we must have

$$m_1\left(\frac{\omega}{2}\right) \overline{m}_0\left(\frac{\omega}{2}\right) + m_1\left(\frac{\omega+1}{2}\right) \overline{m}_0\left(\frac{\omega+1}{2}\right) = 0 \forall \omega \in [0,1] \tag{12}$$

For $\omega \in [0,1/2]$, replacing ω by 2ω in the above equation yeilds

$$\left(m_1\overline{m}_0 + T_{1/2}\left(m_1\overline{m}_0\right)\right) = 0$$

on $[0,1/2]$. A similar argument shows that (12) holds also on $[-1/2,0]$. The periodicity of m_1 and m_0 yeilds the result on all of $\mathbb{R}$. ■

Definition 11 *(the wavelet filter)*

Suppose $\{V_j\}_{j\in\mathbb{Z}}$ *is an MRA in* $L^2(\mathbb{R})$ *with associated scaling function* φ. *Any solution* m_1*of (11) is called the dual auxiliary function and its Fourier coefficients* $\{g_k\}_{k\in\mathbb{Z}}$ *given by (9) is called the wavelet filter. The function* ψ *defined by (10) is called the wavelet (or the morher wavelet) determined by the MRA.*

The following lemma gives a family of solutions of (11).

Lemma 12 *A class of solutions of (11) is given by the formula*

$$m_1 = QT_{1/2}\overline{m}_0, \tag{13}$$

where $Q \in L^2(0,1)$ *with* $|Q| = 1$ *and* $Q + T_{1/2}Q = 0$.

If we choose

$$Q(\omega) = e^{2\pi i\omega}$$

then (13) becomes

$$\begin{aligned}
\sum_n g_n e^{2\pi in\omega} &= e^{2\pi i\omega}\sum_n \overline{h}_n e^{-2\pi in(\omega+1/2)} \\
&= e^{2\pi i\omega}\sum_n (-1)^n \overline{h}_n e^{-2\pi in\omega} \\
&= \sum_n (-1)^n \overline{h}_n e^{-2\pi i(n-1)\omega} \\
&= \sum_n (-1)^{n-1} \overline{h}_{1-n} e^{2\pi in\omega}
\end{aligned}$$

which holds identically on $[0,1/2]$. This results in the relations

$$g_n = (-1)^{n-1}\overline{h}_{1-n}\ \forall n \in \mathbb{Z}.$$

Corollary 13 $m_1(0) = 0$.

Proof. For $\omega = 0,\ m_1(0)\overline{m}_0(0) + m_1\left(\frac{1}{2}\right)\overline{m}_0\left(\frac{1}{2}\right) = 0$. Since $\overline{m}_0(0) = 1, \overline{m}_0\left(\frac{1}{2}\right) = 0$,we get $m_1(0) = 0$. ■

Corollary 14 $\widehat{\psi}(0) = 0$.

Proof. Follows immediately from equation (10) and the above corollary. ■

It now follows that $\int \psi = 0$, i.e., ψ is integrable, and, by the Reimann-Lebesgue Lemma, $\left|\widehat{\psi}\right| \to 0$ as $|\omega| \to \infty$. In other words, ψ is a band-pass filter.

3.2 Wavelet Construction

To construct a wavelet ψ, a suitable 2π periodic function G satisfying Theorem 10 has to be chosen.

We show next that this choice of ψ produces an orthonormal basis for W_0.

Lemma 15 *Let $f \in V_1$ and define $F_0(\omega) := \sum_n f_n^0 e^{2\pi i\omega}$, where $f_n^0 = \langle f, \varphi_{0,n}\rangle$ and $F_1(\omega) := \sum_n f_n^1 e^{2\pi i\omega}$, where $f_n^1 = \langle f, \varphi_{1,n}\rangle$. Then*

$$D_1 F_0 = F_1 \overline{m}_0 + T_{1/2}(F_1 \overline{m}_0)$$

on $[0,1/2]$.

Proof. Since $f = \sum \langle f, \varphi_{n,1}\rangle \varphi_{1,n} = \sum f_n^1 D_1 T_n \varphi$,

$$\begin{aligned}
\widehat{f} &= \sum f_n^1 D_{-1} e_n \widehat{\varphi} \\
&= D_{-1}\widehat{\varphi} \sum f_n^1 e_n \\
&= D_{-1} F_1 \widehat{\varphi}.
\end{aligned}$$

Hence,

$$\begin{aligned}
f_n^0 &= \langle f, \varphi_{0,n}\rangle = \langle f, T_n\varphi\rangle = \left\langle \widehat{f}, e_n\widehat{\varphi}\right\rangle \\
&= \int_{-\infty}^{\infty} e^{-2\pi i n\omega} \widehat{f}(\omega)\, \overline{\widehat{\varphi}}(\omega)\, d\omega \\
&= \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-2\pi i n\omega} F_1\left(\frac{\omega}{2}\right) \widehat{\varphi}\left(\frac{\omega}{2}\right) \overline{\widehat{\varphi}}(\omega)\, d\omega \\
&= \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-2\pi i n\omega} F_1\left(\frac{\omega}{2}\right) \overline{m}_0\left(\frac{\omega}{2}\right) \left|\widehat{\varphi}\left(\frac{\omega}{2}\right)\right|^2 d\omega \\
&= \frac{1}{\sqrt{2}} \int_0^1 e^{-2\pi i n\omega} \sum_k F_1\left(\frac{\omega+k}{2}\right) \overline{m}_0\left(\frac{\omega+k}{2}\right) \left|\widehat{\varphi}\left(\frac{\omega+k}{2}\right)\right|^2 d\omega \\
&= \int_0^1 e^{-2\pi i n\omega} \left(F_1\left(\frac{\omega}{2}\right) \overline{m}_0\left(\frac{\omega}{2}\right) + F_1\left(\frac{\omega+1}{2}\right) \overline{m}_0\left(\frac{\omega+1}{2}\right)\right) d\omega.
\end{aligned}$$

Therefore,

$$\sqrt{2} F_0(\omega) = F_1\left(\frac{\omega}{2}\right) \overline{m}_0\left(\frac{\omega}{2}\right) + F_1\left(\frac{\omega+1}{2}\right) \overline{m}_0\left(\frac{\omega+1}{2}\right) \quad \forall \omega \in [0,1]$$

For $\omega \in [0,1/2]$ we may replace ω by 2ω in the above equation to obtain

$$D_1 F_0 = F_1 \overline{m}_0 + T_{1/2}(F_1 \overline{m}_0)$$

on $[0,1/2]$. ■

Theorem 16 *If (13) holds, then the set $\{T_n\psi\}_{n\in\mathbb{Z}}$ is an ONB for W_0.*

Proof. We calculate

$$\begin{aligned}
\sum_n \left|\widehat{\psi}\,(\omega+n)\right|^2 &= \sum_n \left|\left(m_1\left(\frac{\omega+n}{2}\right)\widehat{\varphi}\left(\frac{\omega+n}{2}\right)\right)\right|^2 \\
&= \left|m_1\left(\frac{\omega}{2}\right)\right|^2 + \left|m_1\left(\frac{\omega+1}{2}\right)\right|^2 \\
&= \left|\overline{m}_0\left(\frac{\omega+1}{2}\right)\right|^2 + \left|\overline{m}_0\left(\frac{\omega}{2}\right)\right|^2 \\
&= \left|m_0\left(\frac{\omega+1}{2}\right)\right|^2 + \left|m_0\left(\frac{\omega}{2}\right)\right|^2 = 1,
\end{aligned}$$

where we used equation (13) to write $|m_1| = T_{1/2}\,|H|$. This shows that $\{T_n\psi\}_{n\in\mathbb{Z}}$ is an ONB.

Next we show that $\{T_n\psi\}_{n\in\mathbb{Z}}$ is complete in W_0. Suppose $f \in W_0$ is orthogonal to $\{T_n\psi\}_{n\in\mathbb{Z}}$. Then $f \in V_1 \ominus V_0$. Therefore, $F_0 = 0$. By Lemma 15,

$$F_1\overline{m}_0 = -T_{1/2}F_1\overline{m}_0. \tag{14}$$

On the other hand we can show that

$$0 = \langle f, T_n\psi\rangle = \frac{1}{\sqrt{2}}\left(F_1\overline{m}_1 + T_{1/2}\left(F_1\overline{m}_1\right)\right),$$

giving

$$F_1\overline{m}_1 = -T_{1/2}\left(F_1\overline{m}_1\right). \tag{15}$$

Finally,since we also have

$$m_1\overline{m}_0 = -T_{1/2}m_1\overline{m}_0, \tag{16}$$

multiplying (14) by the conjgate of (16),

$$\begin{aligned}
F_1\overline{m}_1\,|m_0|^2 &= T_{1/2}\left(F_1\overline{m}_1\,|m_0|^2\right) \\
&= T_{1/2}\left(F_1\overline{m}_1\right)T_{1/2}\,|m_0|^2 \\
&= -F_1\overline{m}_1T_{1/2}\,|m_0|^2.
\end{aligned}$$

Since $|m_0|^2 + T_{1/2}\,|m_0|^2 = 1$,

$$F_1\overline{m}_1 = 0.$$

Similarly,

$$F_1\overline{m}_0 = 0.$$

Hence,

$$\begin{aligned}
0 &= |F_1\overline{m}_0|^2 + |F_1\overline{m}_1|^2 \\
&= |F_1|^2\left(|\overline{m}_0|^2 + |\overline{m}_1|^2\right).
\end{aligned}$$

Therefore, $F_1 = 0$ on $[0, 1/2]$. Consequently $f_n^1 = 0$ for all $n \in \mathbb{Z}$. Hence, $f = 0$ since $f \in V_1$. ■