Two Dimensional Multiresolution Approximations

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1 Introducrion

A multiresolution approximation for the space $L^2(\mathbb{R}^2)$ can be defined in the same way as for the one dimensional case. In the most general situation, the space variable and the scale variable are both points in $\mathbb{R}^2$. Such a general definition, however, is seldomely used in practice and special extesions from the one dimensional case are used in stead. One way to define such an extension is to start with an MRA $\{V_j\}_{j\in\mathbb{Z}}$ for $L^2(\mathbb{R})$, with scaling function φ and wavelet ψ and define an MRA for $L^2(\mathbb{R}^2)$ by means of the orthonormal basis

$$\left\{\psi_{j_1,n_1}(x_1)\,\psi j_{2,n_2}(x_2)\right\}_{n_1,j_1,n_2,j_2\in\mathbb{Z}}.$$

This definition has the drawback of mixing information form two resolutions j_1, j_2 which need not be the same resolution. The MRA to be developed in this chapter avoids this drawback by defining a class of seperable wavelets whose elements are products of one dimensional wavelets dilated at the same resolution.

2 Degenerate representations

A function $f \in L^2(\mathbb{R}^2)$ is called degenerate if

$$f(x,y) = \sum_m f_m(x)\, g_m(y),$$

where the sum is finite, $f_m, g_m \in L^2(\mathbb{R})$.

The fact that the set of degenerate functions is dense in $L^2(\mathbb{R}^2)$ makes it possible to obtain seperable multiresolution analysis for $L^2(\mathbb{R}^2)$. Given two closed subspaces V, W of $L^2(\mathbb{R})$ we will use the notation $V \otimes W$ to denote the closed subspace of $L^2(\mathbb{R}^2)$ generated by degenerate representations of the elements of V and W.

Thanks to this density, we have $L^2(\mathbb{R}^2) = L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ and the following theorem.

Theorem 1 *Let V, W be two closed subspaces of $L^2(\mathbb{R})$. If $\{\eta_k\}_{k\in\mathbb{Z}}$ is an ONB for V and $\{\xi_k\}_{k\in\mathbb{Z}}$ is an ONB for W then $\left\{\eta_{k_1}\xi_{k_2}\right\}_{k_1,k_2\in\mathbb{Z}}$ is an ONB for $V \otimes W$.*

The operation $V \otimes W$ is called the tensor product of V and W. The basic property of this tensor product is the following distributivity property

$$(U+V)\otimes W = U\otimes W + V\otimes W.$$

(This relation can easily be shown by demonstrating that $f \perp (U+V) \otimes W$ if and only if $f \perp U \otimes W + V \otimes W$.)

It follows from this property that

$$(V_1 + V_2) \otimes (W_1 + W_2) = V_1 \otimes W_1 + V_1 \otimes W_2 + V_2 \otimes W_1 + V_2 \otimes W_2.$$

3 Seperable wavelet bases

Let $\{V_j\}_{j\in\mathbb{Z}}$ be an MRA for $L^2(\mathbb{R})$ with scaling function φ and corresponding wavelet ψ. A seperable MRA for $L^2(\mathbb{R}^2)$ is obtained by taking

$$V_j^2 = V_j \otimes V_j.$$

It can be shown that $\{V_j^2\}_{j\in\mathbb{Z}}$ is an MRA for $L^2(\mathbb{R}^2)$ with scaling function ϕ^2 where

$$\phi^2(x,y) = \varphi(x)\varphi(y).$$

By Theorem 1, $\{\tau_n \phi^2\}_{n\in\mathbb{Z}^2}$ is an othonormal basis for V_0^2. Observe that we use the notation $n = (n_1, n_2)$. Denoting

$$\phi_{j,n}^2 = \sqrt{2^j} D_j \tau_n \phi^2$$

we obtain the orthonormal basis $\{\phi_{j,n}^2\}_{n\in\mathbb{Z}^2}$ for V_j^2. In order to arrive at a wavelet basis we consider the decompositoin

$$\begin{aligned} V_j^2 \oplus W_j^2 &= V_{j+1}^2 \\ &= V_{j+1} \otimes V_{j+1} \\ &= (V_j \oplus W_j) \otimes (V_j \oplus W_j) \\ &= V_j \otimes V_j \oplus V_j \otimes W_j \oplus W_j \otimes V_j \oplus W_j \otimes W_j. \end{aligned}$$

Since $V_j^2 = V_j \otimes V_j$, we see that

$$W_j^2 = V_j \otimes W_j \oplus W_j \otimes V_j \oplus W_j \otimes W_j.$$

Now define the functions $\psi^1, \psi^2, \psi^3 \in L^2(\mathbb{R}^2)$ by

$$\begin{aligned} \psi^1 &= \varphi \otimes \psi, \\ \psi^2 &= \psi \otimes \varphi, \\ \psi^3 &= \psi \otimes \psi. \end{aligned}$$

Then $\{\tau_n\psi^1, \tau_n\psi^2, \tau_n\psi^3\}_{n\in\mathbb{Z}^2}$ is an orthonormal basis for W_0^2. Letting

$$\begin{aligned} \psi_{j,n}^1 &= \sqrt{2^j} D_j \tau_n \psi^1, \\ \psi_{j,n}^2 &= \sqrt{2^j} D_j \tau_n \psi^2, \\ \psi_{j,n}^3 &= \sqrt{2^j} D_j \tau_n \psi^3, \end{aligned}$$

we obtain the orthonormal basis $\{\psi_{j,n}^1, \psi_{j,n}^2, \psi_{j,n}^3\}_{n\in\mathbb{Z}^2}$ for W_j^2.

It can be shown that

$$L^2(\mathbb{R}^2) = \bigoplus_{j\in\mathbb{Z}} W_j^2,$$

from which we have the following theorem.

Theorem 2 $\left\{\psi^1_{j,n}, \psi^2_{j,n}, \psi^3_{j,n}\right\}_{n\in\mathbb{Z}^2, j\in\mathbb{Z}}$ *is an orthonormal basis for* $L^2\left(\mathbb{R}^2\right)$.

Furthermore,

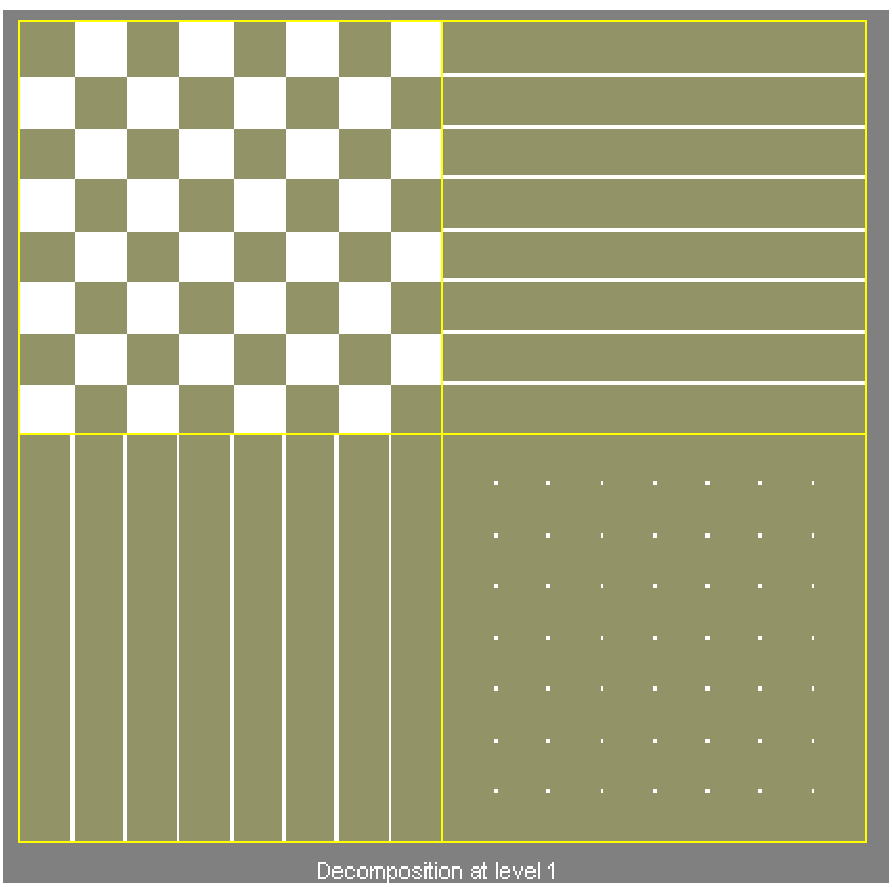
$$L^2\left(\mathbb{R}^2\right) = V_0^2 \bigoplus_{j\geq 0} W_j^2.$$

This decomposes $L^2\left(\mathbb{R}^2\right)$ into an approximation space V_0^2 and detail spaces $W_j^2, j \geq 0$. The three wavelets ψ^1, ψ^2, ψ^3 extract image information at different scales and directions. Since

$$\widehat{\psi}^1\left(\omega_1, \omega_2\right) = \widehat{\varphi}\left(\omega_1\right)\widehat{\psi}\left(\omega_2\right),$$

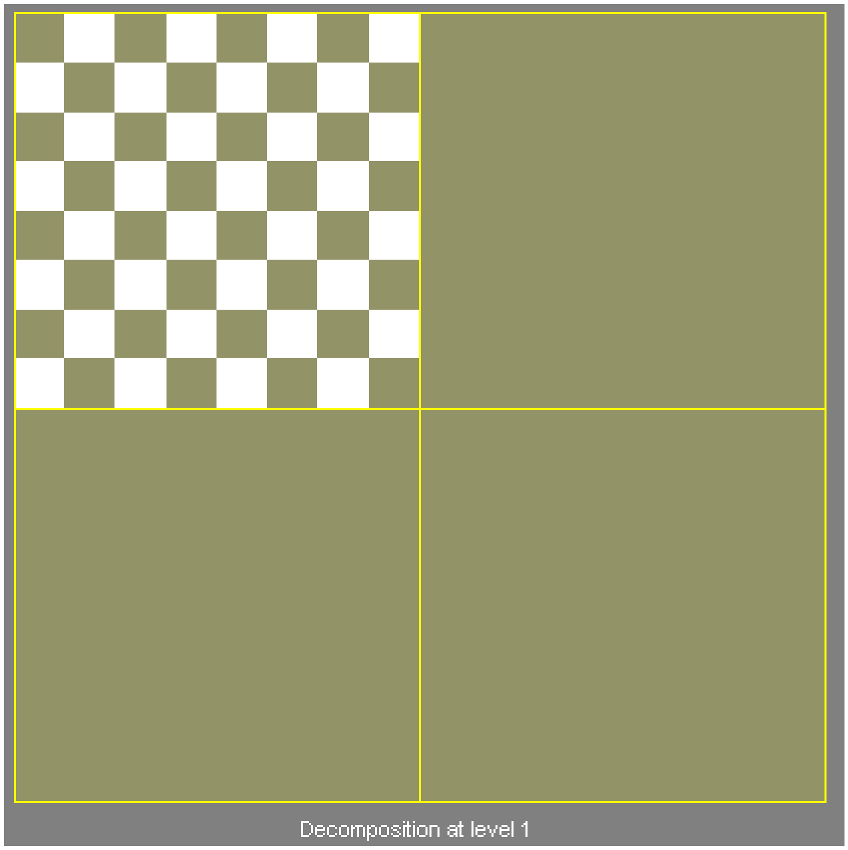
$\left|\widehat{\psi}^1\right|$ is high at low horizontal frequencies and high vertical frequencies. Thus ψ^1 detects rapid vertical changes or horizontal edges. Similarly, ψ^2 detects rapid horizontal changes or vertical edges and ψ^3 detects rapid diagonal changes or corners.

Example 3 *The following figure shows the wavelet analysis of a chess board. The wavelet used in the analysis is db2 which has two vanishing moments. The figure shows the first level approximation of the chess board, the horizontal, vertical and diagonal details.*



Edge and corner detection by smooth wavelets

The following figure shows the analysis of the chess board with the Haar wavelet. Because the Haar wavelet is not smooth, no edges or corners were detected.



Analysis with the Haar wavelet

The approximation projections $P_j : L^2(\mathbb{R}^2) \to V_j^2$ are given by

$$P_j f = \sum_{n\in\mathbb{Z}^2} c_n^j \phi_{j,n}^2$$

where

$$\begin{aligned} c_n^j &= \left\langle f, \phi_{n,j}^2 \right\rangle \\ &= \sqrt{2^j} \int\int f(x,y)\,\overline{\varphi}_{j,n_1}(x)\,\overline{\varphi}_{j,n_2}(y)\,dydx \\ &= \sqrt{2^j} \int \left(\int f(x,y)\,\overline{\varphi}_{j,n_2}(y)\,dy \right) \overline{\varphi}_{j,n_1}(x)\,dx. \end{aligned}$$

Hence, the approximation in the space V_j^2 is performed by projecting the image $f(x,y)$ in the vertical direction first and then in the horizontal direction (or vice versa). In terms of the discrtization operators, we can restate the above transformations as

$$c^j = A_j f = \sqrt{2^j} A_j^x A_j^y f.$$

In a similar fashion the detail projections $Q_j : L^2(\mathbb{R}^2) \to V_j^2$ are given by

$$Q_j f = \sum_{n\in\mathbb{Z}^2} d_n^{1,j} \psi_{j,n}^1 + d_n^{2,j} \psi_{j,n}^2 + d_n^{3,j} \psi_{j,n}^3,$$

where

$$\begin{aligned} d^{1,j} &= B_j^1 f = \sqrt{2^j} A_j^x B_j^y f, \\ d^{2,j} &= B_j^2 f = \sqrt{2^j} B_j^x A_j^y f, \\ d^{3,j} &= B_j^3 f = \sqrt{2^j} B_j^x B_j^y f. \end{aligned}$$

The decomposition

$$V_{j+1}^2 = V_j^2 \oplus W_j^2$$

implies the reconstruction formula

$$P_{j+1} f = P_j f + Q_j f.$$

4 The Decomposition and reconstruction operators

To arrive at a representation for the decomposition and the reconstruction operators in the two dimensional case we need the following notation. Let $h = (h_n)$ and $g = (g_n)$ denote as ususal the scaling and wavelet filters associated with scaling function φ and wavelet function ψ respectively. Define the horizontal convolution operator H_x and the veritcal convolution operator H_y on $\ell^2(\mathbb{Z}^2)$ by

$$\begin{aligned} (H_x c)_n &= \left\langle c_{(n_1,\cdot)}, \tau_{n_2} h \right\rangle = c_{(n_1,\cdot)} * \rho \overline{h}\,(n_2)\,, \\ (H_y c)_n &= \left\langle c_{(\cdot,n_2)}, \tau_{n_1} h \right\rangle = c_{(\cdot,n_2)} * \rho \overline{h}\,(n1)\,. \end{aligned}$$

In a similar fashion we define the horizontal and vertical operators G_x, G_y, G_x^1, G_y^1 by

$$\begin{aligned} (G_x c)_n &= \left\langle c_{(n_1,\cdot)}, \tau_{n_2} g \right\rangle = c_{(n_1,\cdot)} * \rho \overline{g}\,(n_2)\,, \\ (G_y c)_n &= \left\langle c_{(\cdot,n_2)}, \tau_{n_1} g \right\rangle = c_{(\cdot,n_2)} * \rho \overline{g}\,(n_1)\,, \end{aligned}$$

We then have the following theorem.

Theorem 4 *With δ_1 the down sampling operator and δ_{-1} the upsampling operators as before, we have the following relations.*

$$\begin{aligned} A_j f &= \delta_1 H_x \delta_1 H_y A_{j+1} f, \\ B_j^1 f &= \delta_1 H_x \delta_1 G_y A_{j+1} f, \\ B_j^2 f &= \delta_1 G_x \delta_1 H_y A_{j+1} f, \\ B_j^3 f &= \delta_1 G_x \delta_1 G_y A_{j+1} f \end{aligned}$$

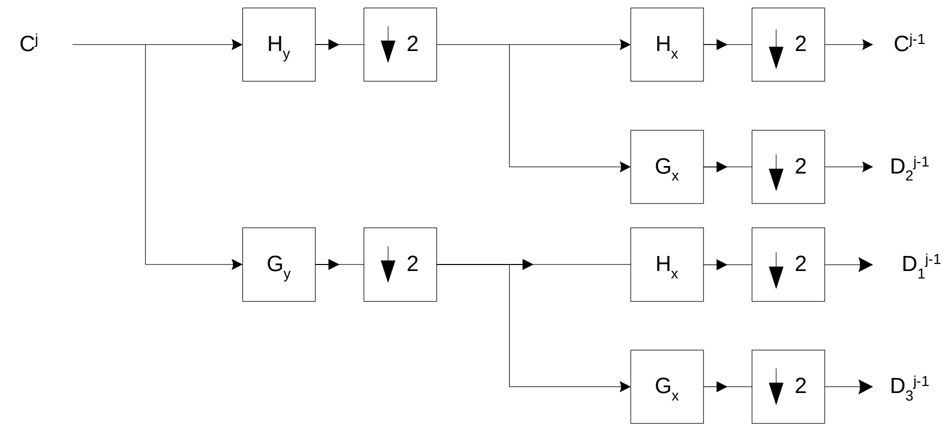
at the decomposition level, and

$$A_{j+1} f = H_y^* \delta_{-1} H_x^* \delta_{-1} A_j f + G_y^* \delta_{-1} H_x^* \delta_{-1} B_j^1 f + H_y^* \delta_{-1} G_x^{1*} \delta_{-1} B_j^2 f + G_y^* \delta_{-1} G_x^* \delta_{-1} B_j^3 f$$

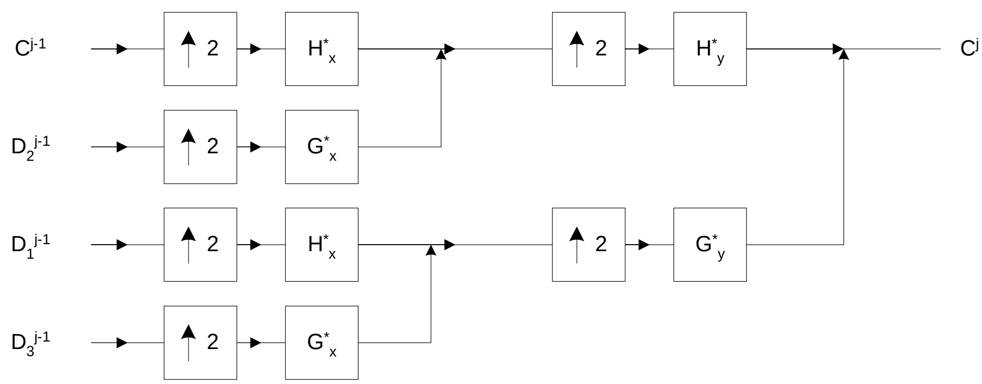
at the reconstruction level.

5 The fast 2-dimensional wavelet transform

The fast wavelet transform and its inverse are based on the recursions in Theorem 4 and are shown in the following two diagrams.



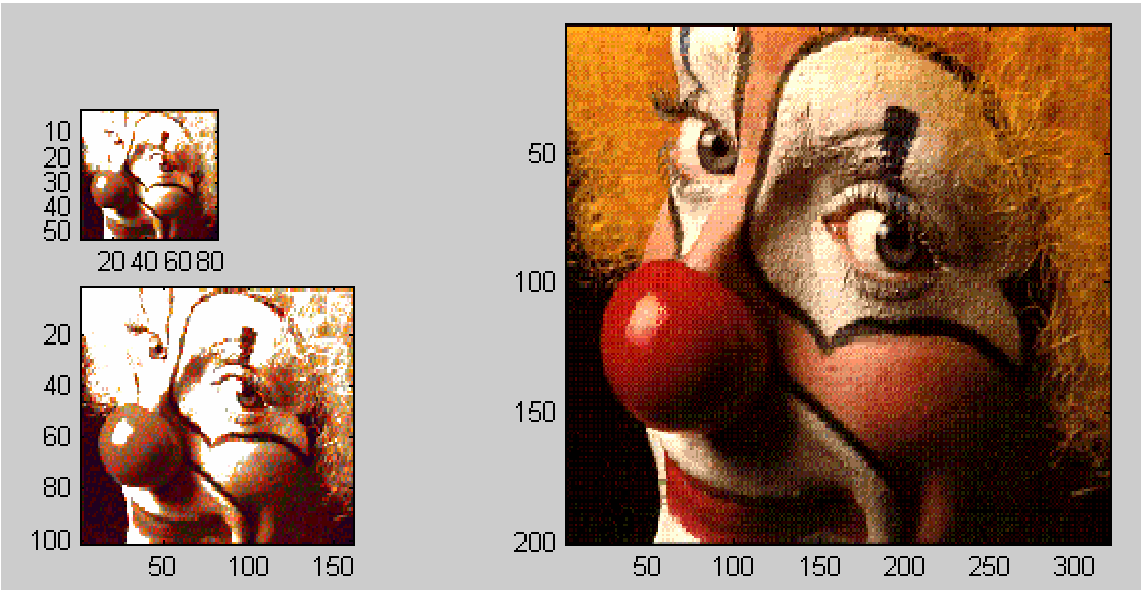
The fast wavelet transform in 2-D



The inverse wavelet transform in 2-D

Notice that the decomposed approximation and details each has 1/4 the sample size of that of the previous higeher resolution. Thus, the decompositoin does not increase the original sample size. It can be shown that for finite images of size N, the decomposition and reconstruction stages require $O\left(N^2\right)$ additions and multiplications to perform.

Example 5 *The following figure shows the original and the first two lwvels of approximation of the image of the clown. The original image has size* 320×200. *The sizes of the first and second level of approximation are* 160×100 *and* 80×50, *respectivley. Notice that image is still visually recognizable after these two levels of approxiamation. This observation is the basis for image compression to be dealt with later.*



The first two levels of approxiamtion (left) of the image of the clown (right)