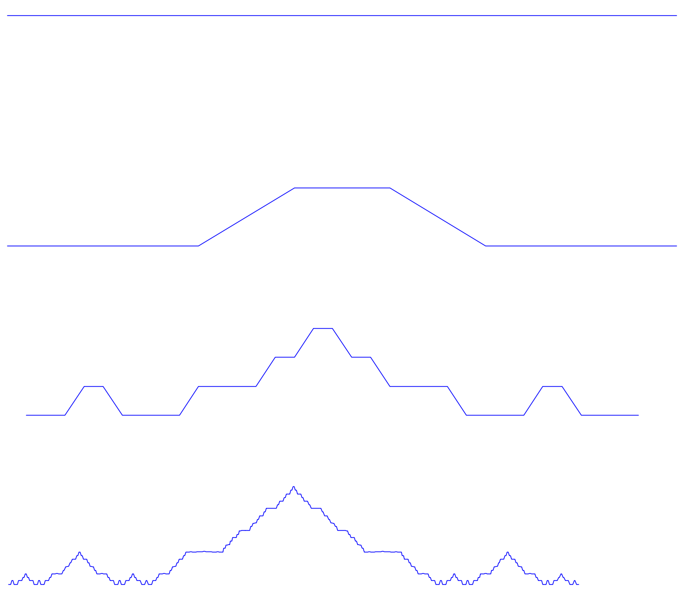
II.3 MULTIFRACTALS

1. Fractal sets and self similar functions

Let $S \subset \mathbb{R}^n$ be a bounded set. S is called self similar if there exist disjoint sets $S_1, S_2, \ldots S_k$ and affine transformations $K_1, K_2, \ldots, K_k$ representing scaling, translation or rotation such that $S_i = K_i S$ and $S = \cup_{i=1}^{k} S_i$.

Example 1.1. The von Kotch curve



$$l_k = \left(\frac{4}{3}\right)^k.$$

Example 1.2. The Cantor set

$$l_k = \left(\frac{2}{3}\right)^k.$$

2. Fractal Dimension

Definition 2.1. *(capacity dimension)*

Let $S \subset \mathbb{R}^n$ be a bounded set. The capacity dimension of S is defined as

$$D = -\liminf_{a \to 0} \frac{\log N(a)}{\log a},$$

where $N(a)$ is the number of balls in $\mathbb{R}^n$ of radius a needed to cover S.

Example 2.2. For the von Kotch curve, since $l_k = \left(\frac{4}{3}\right)^k$, we need 4^k balls of radius $a_k = \frac{1}{3^k}$ to cover l_k. Thus

$$\frac{\log N(a_k)}{\log a_k} = \frac{\log 4}{\log \frac{1}{3}} = -\frac{\log 4}{\log 3}$$

and

$$D = \frac{\log 4}{\log 3} > 1.$$

Example 2.3. For the Cantor set, we need 2^kballs of radius $a_k = \frac{1}{3^k}$ to cover l_k. In this case we easily find $D = \frac{\log 2}{\log 3} < 1$.

Definition 2.4. *(the measure of a fractal set)*
Suppose the set S has fractal dimension D. Then its measure M is defined by

$$M = \limsup_{a \to 0} N(a)\, a^D.$$

Roughly speaking, if a set S has fractal dimension D, then the number of balls of radius a needed to cover S is proportional to a^{-D}. That is,

$$N(a) \sim a^{-D}$$

for sufficiently small a.

Self-similar functions Let's define the special affine transformation $\mathcal{A}$ on $L^2(\mathbb{R})$ by

$$\mathcal{A}f(t) = c + pf(l(t-r))$$

where c, p are complex numbers and l, r are real numbers with $l > 0$. The tuplet $(c, p; l, r)$ will be called the associated tuplet.

Definition 2.5. *(self-similar functions)*
Let $f \in L^2(\mathbb{R})$. f is called self-similar on a set S if there exists an affine transformation $\mathcal{A}$ such that

$$\mathcal{A}f(t) = f(t) \;\; \forall t \in S.$$

f is called self-similar if its domain can be partitioned into a finite number of disjoint sets $S_1, S_2, \ldots S_k$ such that f is self-similar on each S_i, $i = 1, 2, \ldots, k$.

Lemma 2.6. *(invariance of affine transformations wrt wavelet transforms)*

Let $\mathcal{A}$ be an affine transformation on $L^2(\mathbb{R})$ with associated tuplet $(c, p; l, r)$ and let $\mathcal{A}'$ be the affine transformation on $L^2_w(\mathbb{R}^+ \times \mathbb{R})$ with associated tuplet $\left(0, \frac{p}{\sqrt{l}}; l, (0, r)\right)$. Then

$$W_\psi \mathcal{A} = \mathcal{A}' W_\psi.$$

Proof. For $f \in L^2(\mathbb{R})$,

$$\begin{aligned} W_\psi \mathcal{A} f(a,b) &= \int_{-\infty}^{\infty} (c + pf(l(t-r))) \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right) dt \\ &= \frac{p}{\sqrt{a}} \int_{-\infty}^{\infty} f(l(t-r)) \psi\left(\frac{t-b}{a}\right) dt. \end{aligned}$$

Using the change of variable $\tau = l(t-r)$ we get

$$\begin{aligned} W_\psi \mathcal{A} f(a,b) &= \frac{p}{l\sqrt{a}} \int_{-\infty}^{\infty} f(\tau) \psi\left(\frac{\tau - l(b-r)}{la}\right) d\tau \\ &= \frac{p}{\sqrt{l}} W_\psi f(la, l(b-r)) \\ &= \frac{p}{\sqrt{l}} W_\psi f(l((a,b) - (0,r))) \\ &= \mathcal{A}' W_\psi f(a,b). \end{aligned}$$

□

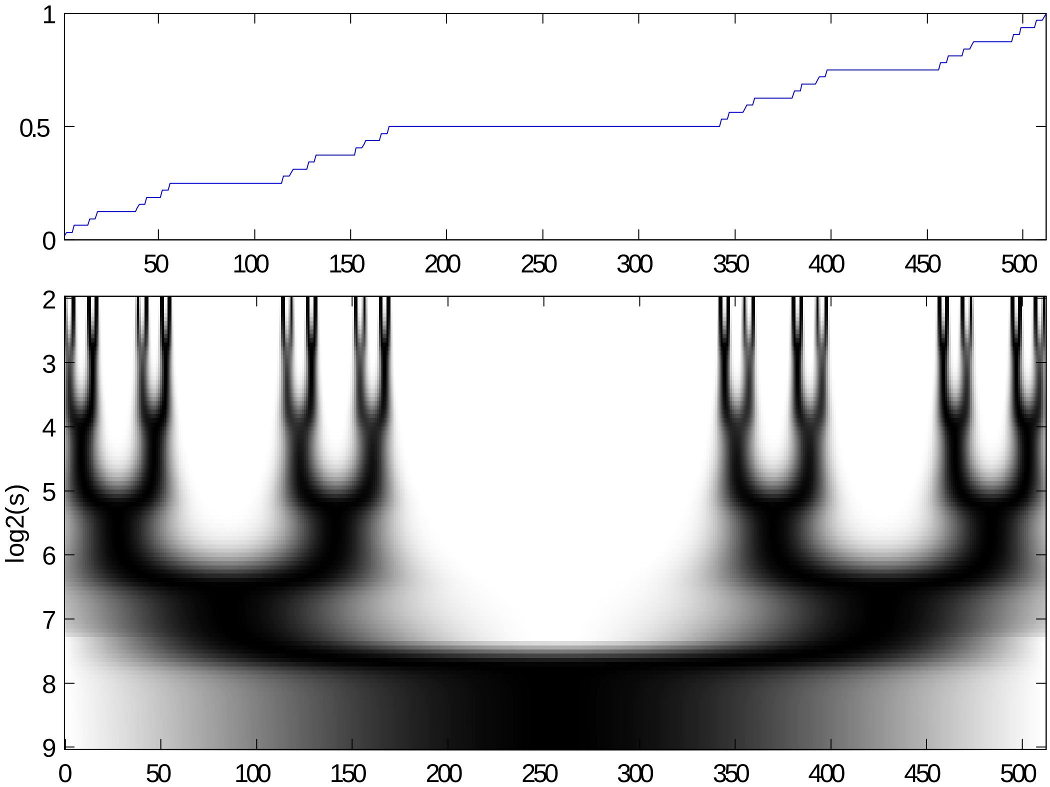
Corollary 2.7. *(invariance of the wavelet transform of self similar sets)*

Suppose the wavelet ψ is supported in $[-C, C]$. If f is self-similar on the interval $S = [\alpha, \beta]$ then $W_\psi f$ is self-similar on $\mathcal{S} = \left[0, \frac{\beta-\alpha}{2C}\right] \times [\alpha, \beta]$.

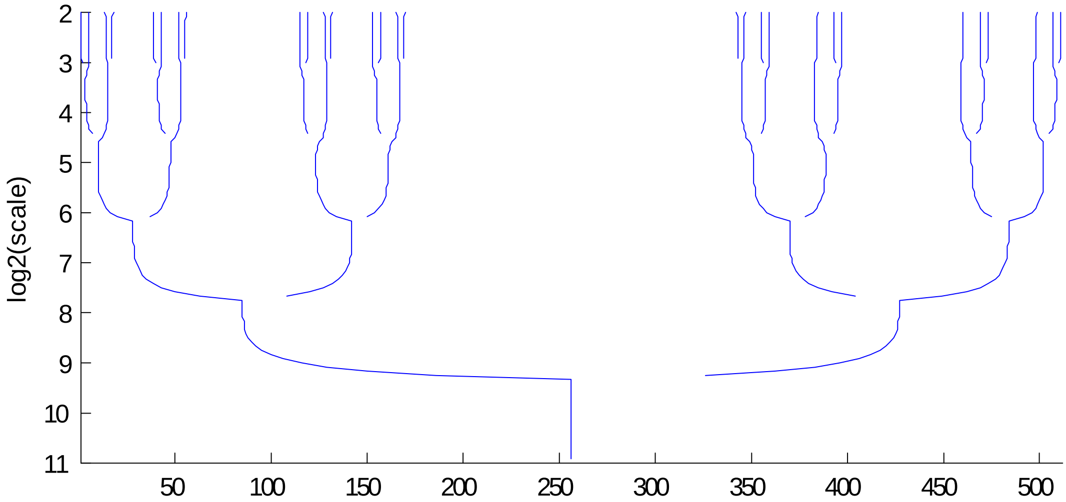
Proof. It is easy to check that, for $(a,b) \in \mathcal{S}$, $\psi_{a,b}$ is supported in S. For $(a,b) \in \mathcal{S}$,

$$\begin{aligned} \mathcal{A}'_i W_\psi f(a,b) &= W_\psi \mathcal{A}_i f(a,b) = \int_{-\infty}^{\infty} \mathcal{A}_i f(t) \psi_{a,b}(t)\, dt \\ &= \int_S \mathcal{A}_i f(t) \psi_{a,b}(t)\, dt = \int_S f(t) \psi_{a,b}(t)\, dt \\ &= \int_{-\infty}^{\infty} f(t) \psi_{a,b}(t)\, dt = W_\psi f(a,b). \end{aligned}$$

□



It follows from the above corollary that if a signal is self similar, then its wavelet transform modulus maxima is also self similar.



3. Singularity spectrum

When a function f has nonisolated α-singularities with possibly varying values of α it is called a multifractal. A multifractal f is said to be homogenous if it has the same α-singularity at all its singular points. The Lipschitz exponent of a multifractal cannot be computed because of the above mentioned variation of the Lipschitz constant α. We develop in this section ways to deal with multifractals. The first step is to partition the domain of a multifractal f into subsets, each associated with a specific value of α. Let

$$S_\alpha := \{t \in \mathbb{R} : f \text{ has an } \alpha\text{-singularity at } t\}.$$

Next we define the dimension function

$$D(\alpha) := \dim S_\alpha,$$

where $\dim S_\alpha$ means the fractal dimension of S_α.

Definition 3.1. *(the spectrum of singularity)*
The spectrum of singularity of a function f is defined to be the support of the dimension function $D(\cdot)$. Here

$$\operatorname{supp} D(\cdot) = \{\alpha : S_\alpha \neq \phi\}.$$

Partition function The wavelet transform modulus maxima can be interpretted as a covering of the singular support of f with domains of wavelets at scale a. Take an $\varepsilon > 0$. For each scale $a > 0$, partition the domain of $W_\psi f(a, \cdot)$ into subintervals $I_p = \left[\alpha_p, \beta_p\right], \; p \in \mathbb{Z}$ of width $a\varepsilon$. In each interval I_p choose b_p such that

$$|W_\psi f(a, b_p)| = \max_{b \in I_p} |W_\psi f(a, b)|$$

The partition function is defined by

$$\mathcal{Z}(q, a) = \sum_p |W_\psi f(a, b_p)|^q \tag{1}$$

In order to determine the decay rate of $\mathcal{Z}(q, a)$ with the scale a ($\mathcal{Z}(q, a) \sim a^{\tau(q)}$) we define

$$\tau(q) = \liminf_{a \to 0^+} \frac{\log \mathcal{Z}(q, a)}{\log a}.$$

Theorem 3.2. *(the decay rate of $\mathcal{Z}$)*
Let $\operatorname{supp} D(\cdot) = \Lambda = [\alpha_{\min}, \alpha_{\max}]$. *Let ψ be a wavelet with $n > \alpha_{\max}$ vanishing moments. If f is self-similar then*

$$\tau(q) = \inf_{\alpha \in \Lambda} \left(q \left(\alpha + \frac{1}{2} \right) - D(\alpha) \right).$$

Proof. (ouline)
For each a, the measure of $S_\alpha \sim a^{-D(\alpha)}$ and $|W_\psi f(a, b)| \sim a^{\alpha + \frac{1}{2}}$. Then

$$\begin{aligned} \mathcal{Z}(q, a) &\sim \int_\Lambda a^{q\left(\alpha + \frac{1}{2}\right)} a^{-D(\alpha)} d\alpha \\ &= \int_\Lambda a^{q\left(\alpha + \frac{1}{2}\right) - D(\alpha)} d\alpha \\ &\leq a^{\tau(q)} \left[\alpha_{\max} - \alpha_{\min}\right]. \end{aligned}$$

□

Proposition 3.3. *(properties of $\tau(\cdot)$ and $D(\cdot)$)*

- **(i):** *$\tau(\cdot)$ is convex increasing.*
- **(ii):** *If f is self similar, then $D(\cdot)$ is convex.*
- **(iii):** *If $D(\cdot)$ is convex then*

$$D(\alpha) = \inf_{q \in \mathbb{R}} \left(q \left(\alpha + \frac{1}{2} \right) - \tau(q) \right). \tag{2}$$

Numerical Calculations

(1) Compute $W_\psi f(a, b)$ and the modulus maxima at each scale a. Chain the maxima across scales.
(2) Compute the partition function $\mathcal{Z}(q, a)$ from (1)
(3) Compute $\tau(q)$ as the slope of the linear relation between $\log \mathcal{Z}(q, a)$ and $\log a$.

(4) Compute $D(\alpha)$ from (2).

Assignment 3: The temperature record from the weather station in Arar (Saudi Arabia) for the period (Jan-1990 to Dec-2006) is posted on the web page in the form of a mat file (tarar.mat). Use this data to compute the fractal dimension of the weather signal and plot your results as a function of q.