1 The Continuous Wavelet Transform

1.1 The Fourier Transform

- We use the terms signal and function interchangeably.

Definition 1 *(The Fourier Transform)*
The Fourier Transform $\widehat{f}$ *of a function* $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ *is defined by*

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i\omega t} dt.$$

If f *is only square integrable, then its Fourier transform is given by*

$$\widehat{f}(\omega) = \lim_{n\to\infty} \int_{-n}^{n} f(t)\, e^{-2\pi i\omega t} dt.$$

- The following elementary operators will be of great use later:
  - The reflection operator $\mathcal{R} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by

    $$\mathcal{R}f(t) = f(-t).$$

  - The translation operator (associated with $b \in \mathbb{R}$) $T_b : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by

    $$T_b f(t) = f(t-b).$$

  - The dilation operator (associated with $a \in \mathbb{R}^+$) $D_a : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by

    $$D_a f(t) = \frac{1}{\sqrt{a}} f\left(\frac{t}{a}\right).$$

  All these operators are unitary; $\mathcal{R}^* = \mathcal{R}$, $T_b^* = T_{-b}$, $D_a^* = D_{1/a}$.

- The Fourier transform has the following elementary properties:
  - $\widehat{\alpha f + \beta g} = \alpha\widehat{f} + \beta\widehat{g}$
  - $\widehat{D_a f} = D_{1/a}\widehat{f}$
  - $\widehat{T_b f} = e^{-2\pi i\omega b}\widehat{f}$

- $\widehat{\mathcal{R}f} = \mathcal{R}\widehat{f}$
- $\widehat{\widehat{f}} = \mathcal{R}\overline{\widehat{f}}$

- We can define an operator $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $\mathcal{F}f = \widehat{f}$. It turns out that $\mathcal{F}$ is a unitary operator, that is,

$$\langle \mathcal{F}f, \mathcal{F}g \rangle = \langle f, g \rangle$$

for all $f, g \in L^2(\mathbb{R})$. This can also be restated as

$$\left\langle \widehat{f}, \widehat{g} \right\rangle = \langle f, g \rangle .$$

In particular, when $f = g$, we get

$$\|\mathcal{F}f\| = \|f\| .$$

- For unitary operators, $\mathcal{F}^{-1} = \mathcal{F}^*$. Thus,

$$f = \mathcal{F}^{-1}\widehat{f} = \mathcal{F}^*\widehat{f}$$

which means

$$f(t) = \int_{-\infty}^{\infty} \widehat{f}(\omega) e^{2\pi i\omega t} d\omega$$

for $\widehat{f} \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and

$$f(t) = \lim_{n\to\infty} \int_{-n}^{n} \widehat{f}(\omega) e^{2\pi i\omega t} d\omega$$

if $\widehat{f}$ is only square integrable.

Definition 2 *(convolution)*
The convolution of two functions $f, g \in L^2(\mathbb{R})$ is a function $f * g$ given by

$$f * g(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau.$$

- We have

$$\widehat{f * g} = \widehat{f}\widehat{g}.$$

Thus $f * g \in L^2(\mathbb{R})$ if and only if $\widehat{f}\widehat{g} \in L^2(\mathbb{R})$. Also,$(f * g) * h = f * (g * h)$ as can be seen by taking the Fourier transform on both sides.

Example (a function whose convolution with itself is not square integrable)

Let $\widehat{f}(\omega) = \begin{cases} \sqrt{n}, & \omega \in \left[n, n + \frac{1}{n^3}\right] \\ 0, & \text{Otherwise} \end{cases}$. Then $\widehat{f} \in L^2(\mathbb{R})$ but $\widehat{f}^2 \notin L^2(\mathbb{R})$.

Therefore, $f * f \notin L^2(\mathbb{R})$.

1.2 The Continuous Wavelet Transform CWT

Definition 3 *(wavelets)*

A wavelet is a function $\psi \in L^2(\mathbb{R})$ with the following properties

(i) $\|\psi\| = 1$,

(ii) $\int_{-\infty}^{\infty} \psi(t)\, dt = 0$,

(iii) $c_\psi := \int_0^\infty \frac{\left|\widehat{\psi}(\omega)\right|^2}{\omega} d\omega < \infty$.

- Condition (i) is merely a normalization condition. Its importance will be clear when whe dicuss wavelet coefficients.
- Condition (ii) implies that $\psi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. It also means that ψ has zero mean value. We can show that $\widehat{\psi}$ is a continuous function.
- The previous comment together with (iii) imply that $\widehat{\psi}(0) = 0$. Observing that $\widehat{\psi}(0) = \int_{-\infty}^{\infty} \psi(t)\, dt$, we see that condition (ii) can be replaced by (iii) and the requirement that $\widehat{\psi}$ is continuous.
- The constant c_ψ is called the admissibility condition. It is necessary to obtain the inverse of the wavelet transform as we shall see later.
- Given a wavelet ψ (also known as a mother wavelet), $a > 0$ and $b \in \mathbb{R}$, we define the dilated and translated version $\psi_{a,b}$ of ψ by

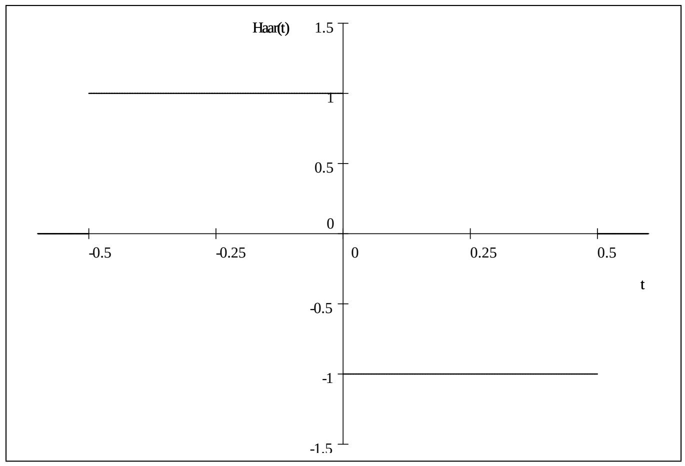
$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right) = T_b D_a \psi(t).$$

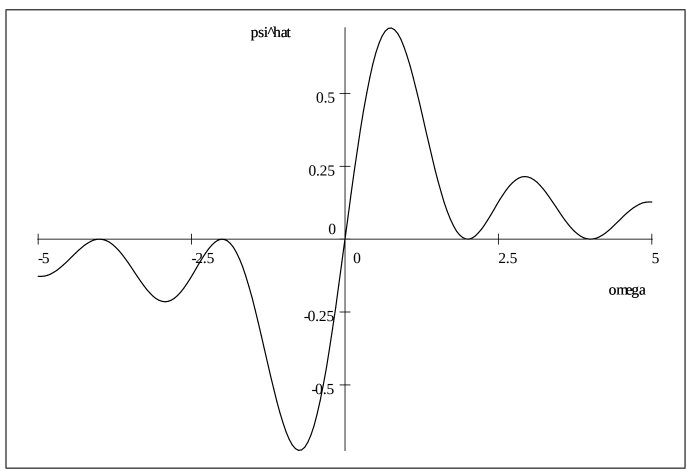
  - $\psi_{a,b}$ is obtained by dilating ψ by a "scale factor" a and shifting the dilated wavelet to the time instant b. The coefficient $\frac{1}{\sqrt{a}}$ is introduced to preserve the normalization of the resulting (child) wavelet $\psi_{a,b}$. Hence,

$$\left\|\psi_{a,b}\right\| = 1 \ \forall (a,b) \in \mathbb{R}^+ \times \mathbb{R}.$$

Examples of wavelets 1. The Haar Wavelet

$$\psi(t) = \chi_{[-1/2,0)}(t) - \chi_{[0,1/2)}(t), \qquad \widehat{\psi}(\omega) = \frac{\sin^2(\pi\omega/2)}{i(\pi\omega/2)}$$

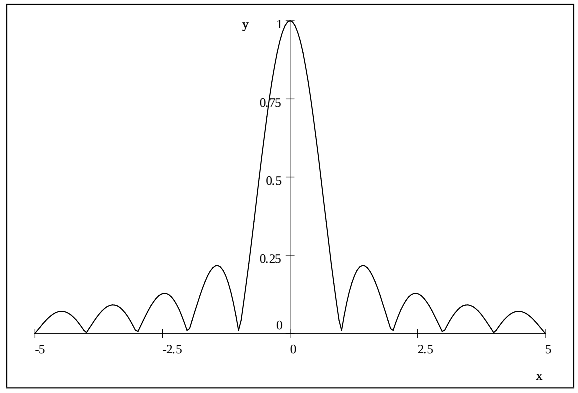


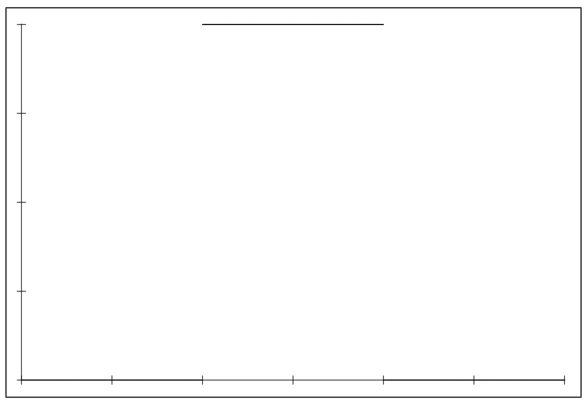


The Haar wavelet has compact support (the interval $[-1/2, 1/2]$) and its Fourier transform has the infinite support $(-\infty, \infty)$.

2. Shannon Wavelets

$$\psi(t) = e^{3\pi i t}\frac{\sin \pi t}{\pi t}, \qquad \widehat{\psi}(\omega) = \chi_{[1,2)}(\omega)$$

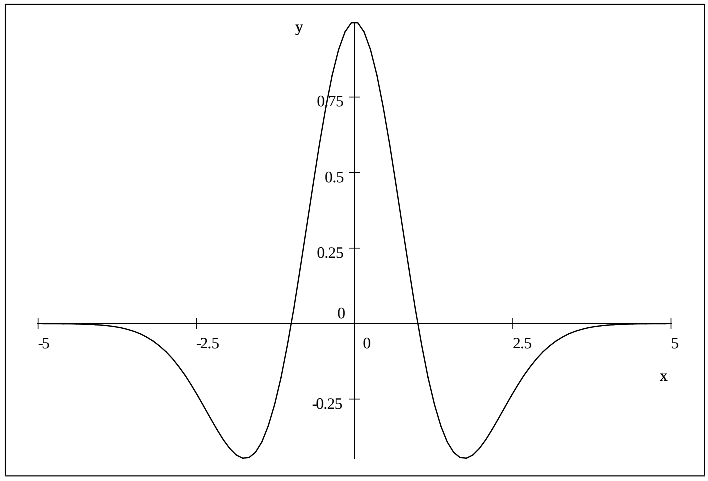


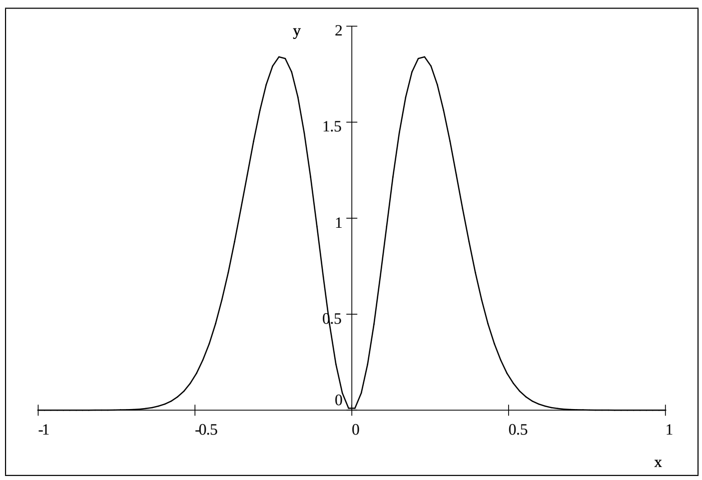


The Shannon wavelet is a complex wavelet with infinite support while its Fourier transform has compact support.

3. The Mexican Hat wavelet

$$\psi(t) = \left(1 - t^2\right) e^{-t^2/2}, \qquad \widehat{\psi}(\omega) = 4\sqrt{2}\pi^{5/2}\omega^2 e^{-2\pi^2\omega^2}$$





The wavelet and its Fourier transform have infinite support but they die out quicly.

Exercise 1 Work out the details for the Fouier transforms of the wavelets in the above examples.

Definition 4 *(The Continuous Wavelet Transform)*

Given a wavelet ψ, the continuous wavelet transform with respect to ψ of a function $f \in L^2(\mathbb{R})$ is the function $\widetilde{f}$ defined by

$$\widetilde{f}(a,b) = \int_{-\infty}^{\infty} f(t) \overline{\frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right)} dt, \ \forall (a,t) \in \mathbb{R}^+ \times \mathbb{R}.$$

- Observe that the CWT is a function of two variables a and b. We can show that $\widetilde{f}$ is square integrable on $\mathbb{R}^+ \times \mathbb{R}$ with respect to the weight function $w(a) = \frac{1}{c_\psi a^2}$. That is

$$\int_{-\infty}^{\infty} \int_{0}^{\infty} \left| \widetilde{f}(a,b) \right|^2 w(a)\, da db < \infty.$$

  We will denote the space of these functions by $L_w^2(\mathbb{R}^+ \times \mathbb{R})$.

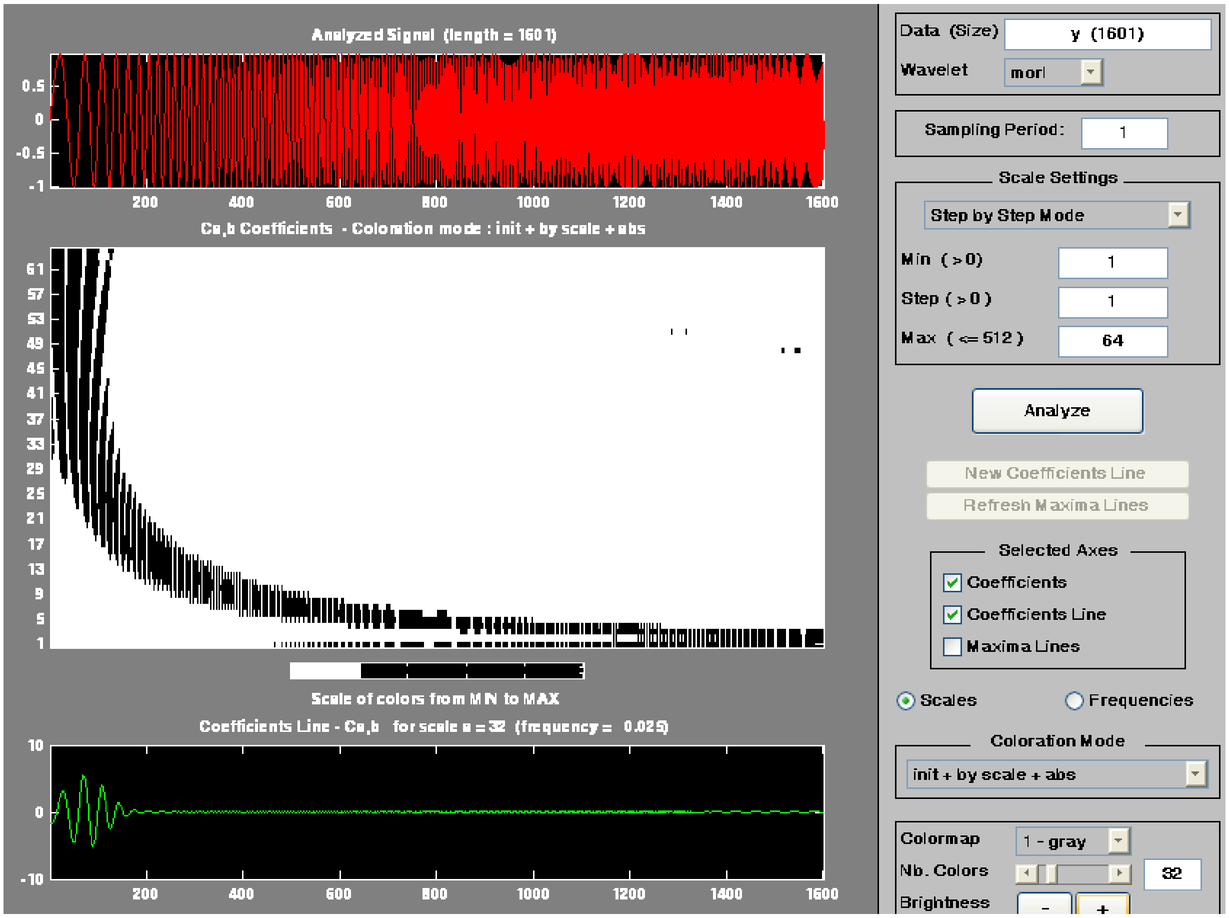
- We can also state the continuous wavelet transform as

$$\widetilde{f}(a,b) = \left\langle f, \psi_{a,b} \right\rangle = f * D_a \mathcal{R} \overline{\psi}(b).$$

  - Thus, the wavelet transform $\widetilde{f}(a,b)$ is the component of f in the direction of the reflected, dilated (by scale a) and shifted (to time instant b) version of ψ.
  - Small values of a (small scales) correspond to rapidly changing modes (high frequencies) and vice versa. Thus, a small value of $\widetilde{f}(a,b)$ indicates weak correlation of f with the shape of the wavelet at scale a and time instant b, whereas a large value of $\widetilde{f}(a,b)$ indicates strong correlation with the shape of the wavelet at scale a and time instant b.

Example (Wavelet Toolbox)

The chirp $f(t) = \sin(2\pi(1+t)t)$, $t \in [0,16]$ is analized using a Morlet wavelet



- We can define a wavelet transform operator $W_\psi : L^2(\mathbb{R}) \to L^2_w(\mathbb{R}^+ \times \mathbb{R})$ by $W_\psi f = \widetilde{f}$. We can show that W_ψ is an isometric operator, that is,

$$\langle W_\psi f, W_\psi g \rangle = \langle f, g \rangle .$$

Its domain is all of $L^2(\mathbb{R})$ but its range is a proper closed subspace M of $L^2_w(\mathbb{R}^+ \times \mathbb{R})$. Thus, $W_\psi^{-1} = W_\psi^*$ on M. In particular, if $f = g$,

$$\|W_\psi f\|^2 = \|f\|^2 .$$

- This gives us the inversion formula (or the reconstruction formula)

$$\begin{aligned} f(t) &= \int_0^\infty \int_{-\infty}^\infty \widetilde{f}(a,b) \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right) w(a)\, dbda \\ &= \int_0^\infty \int_{-\infty}^\infty f * D_a \mathcal{R}\overline{\psi}(b)\, D_a\psi(t-b)\, w(a)\, dbda \\ &= \int_0^\infty f * D_a \mathcal{R}\overline{\psi} * D_a\psi(t)\, w(a)\, da. \end{aligned}$$

To see this, let

$$z(t) = \int_0^\infty f * D_a \mathcal{R}\overline{\psi} * D_a\psi(t)\, w(a)\, da.$$

Taking the Fourier transform on both sides we get

$$\begin{aligned} \widehat{z}(\omega) &= \int_0^\infty \widehat{f}(\omega)\, \widehat{D_a \mathcal{R}\overline{\psi}}(\omega)\, \widehat{D_a\psi}(\omega)\, w(a)\, da \\ &= \widehat{f}(\omega) \int_0^\infty D_{1/a}\mathcal{R}\widehat{\overline{\psi}}(\omega)\, D_{1/a}\widehat{\psi}(\omega)\, w(a)\, da \\ &= \widehat{f}(\omega) \int_0^\infty D_{1/a}\overline{\widehat{\psi}}(\omega)\, D_{1/a}\widehat{\psi}(\omega)\, w(a)\, da \\ &= \widehat{f}(\omega) \int_0^\infty \overline{D_{1/a}\widehat{\psi}(\omega)} D_{1/a}\widehat{\psi}(\omega)\, w(a)\, da \\ &= \widehat{f}(\omega) \int_0^\infty \left|D_{1/a}\widehat{\psi}(\omega)\right|^2 w(a)\, da \\ &= \widehat{f}(\omega) \int_0^\infty a\left|\widehat{\psi}(a\omega)\right|^2 w(a)\, da \\ &= \widehat{f}(\omega) \int_0^\infty \left|\widehat{\psi}(a\omega)\right|^2 \frac{1}{c_\psi a} da. \end{aligned}$$

Using the change of variable $\gamma = a\omega$ gives

$$\widehat{z}(\omega) = \frac{1}{c_\psi} \widehat{f}(\omega) \int_0^\infty \frac{\left|\widehat{\psi}(\gamma)\right|^2}{\gamma} d\gamma = \widehat{f}(\omega).$$

- The explicit form of $W_\psi^* : L_w^2(\mathbb{R}^+ \times \mathbb{R}) \to L^2(\mathbb{R})$ is

$$\left(W_\psi^* g\right)(t) = \int_0^\infty g(a,\cdot) * D_a\psi(t)\, w(a)\, da.$$

- We also have the "reproducing kernel" identity

$$g(a_0, b_0) = \int_{-\infty}^{\infty} \int_{0}^{\infty} K(a_0, b_0, a, b)\, g(a, b)\, w(a)\, da db \;\forall g \in M, \qquad (1)$$

where

$$K(a_0, b_0, a, b) = \left\langle \psi_{a_0, b_0}, \psi_{a,b} \right\rangle.$$

It can be seen by observing that

$$g = W_\psi W_\psi^* g$$

for every g in the range M of the wavelet transform W_ψ.

Exercise 2 Prove the identity (1).

Definition 5 *(the energy density; scalograms)*
The energy density of a function f with respect to the wavelet transform W_ψ at scale a and time b is defined as

$$P_W f(a, b) := |W_\psi f(a, b)|^2.$$

The energy density is called a scalogram. The normalized scalogram is defined as

$$\frac{1}{a} P_W f(a, b).$$

The scaling function The idea behind the scaling function is to accumulate the action of the wavelet transform for large scales (ie., scales $a > a_0$ for some $a_0 > 0$). To do this we write

$$\begin{aligned} f(t) &= \int_0^\infty f * D_a \mathcal{R}\overline{\psi} * D_a \psi(t)\, w(a)\, da \\ &= \int_0^{a_0} f * D_a \mathcal{R}\overline{\psi} * D_a \psi(t)\, w(a)\, da + \int_{a_0}^\infty f * D_a \mathcal{R}\overline{\psi} * D_a \psi(t)\, w(a)\, da \end{aligned}$$

As before putting $z(t) = \int_{a_0}^\infty f * D_a \mathcal{R}\overline{\psi} * D_a \psi(t)\, w(a)\, da$ and taking the Fourier transform we obtain

$$\begin{aligned} \widehat{z}(\omega) &= \widehat{f}(\omega) \int_{a_0}^\infty \left| D_{1/a} \widehat{\psi}(\omega) \right|^2 w(a)\, da \\ &= \frac{1}{c_\psi} \widehat{f}(\omega) \int_{a_0}^\infty \frac{\left| \widehat{\psi}(a\omega) \right|^2}{a} da = \frac{1}{c_\psi} \widehat{f}(\omega) \int_{a_0 \omega}^\infty \frac{\left| \widehat{\psi}(\gamma) \right|^2}{\gamma} d\gamma. \end{aligned}$$

Let ϕ be any function such that

$$\left|\widehat{\phi}(\omega)\right|^2 = \int_\omega^\infty \frac{\left|\widehat{\psi}(\gamma)\right|^2}{\gamma} d\gamma.$$

Then

$$\begin{aligned}\widehat{z}(\omega) &= \frac{1}{a_0 c_\psi}\widehat{f}(\omega)\left|D_{1/a_0}\widehat{\phi}(\omega)\right|^2 \\ &= \frac{1}{a_0 c_\psi}\widehat{f}(\omega)\, \widehat{D_{a_0} R\overline{\phi}}(\omega) D_{a_0}\widehat{\phi}(\omega)\end{aligned}$$

and

$$\begin{aligned}z(t) &= \frac{1}{a_0 c_\psi} f * D_{a_0}\mathcal{R}\overline{\phi} * D_{a_0}\phi(t) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)\frac{1}{\sqrt{a_0}}\overline{\phi}\left(\frac{\tau-b}{a_0}\right)\frac{1}{\sqrt{a_0}}\phi\left(\frac{t-b}{a_0}\right) d\tau db \\ &= \int_{-\infty}^{\infty} f^\dagger(a_0,b)\,\phi_{a_0,b}(t)\, db,\end{aligned}$$

where

$$f^\dagger(a_0,b) = \frac{1}{a_0 c_\psi}\int_{-\infty}^{\infty} f(\tau)\,\overline{\phi}_{a_0,b}(\tau)\, d\tau = \frac{1}{a_0 c_\psi}\left\langle f, \phi_{a_0,b}\right\rangle.$$

Time-Frequency Resolution

Definition 6 *(energy spread of a signal)*
Suppose $f \in L^2(\mathbb{R})$ is a signal. The energy center (t_c, ω_c) of f is defined by

$$\begin{aligned}t_c &= \frac{\int_{-\infty}^{\infty} t\,|f(t)|^2\, dt}{\|f\|^2}, \\ \omega_c &= \frac{\int_{-\infty}^{\infty} \omega\left|\widehat{f}(\omega)\right|^2 d\omega}{\left\|\widehat{f}\right\|^2},\end{aligned}$$

respectively. The energy spread of f is defined to be the box in the t-ω plane centered at (t_c, ω_c) with spread σ_t in the t-direction and σ_ω in the ω-direction, where

$$\begin{aligned} \sigma_t^2 &= \int_{-\infty}^{\infty} (t - t_c)^2 \left| f(t) \right|^2 dt, \\ \sigma_\omega^2 &= \int_{-\infty}^{\infty} (\omega - \omega_c)^2 \left| \widehat{f}(\omega) \right|^2 d\omega. \end{aligned}$$

- The Heisenberg uncertainty principle: Suppose ψ is a wavelet with spreads σ_t, σ_ω. It can be shown that the spreads of $\psi_{a,b}$ are $a\sigma_t$ and σ_ω / a. Therefore, in all cases the size of the energy spread is $\sigma_t \sigma_\omega$. This shows that by controlling the scale we can improve the time resolution of the wavelet or the frequency resolution but not both.

Real wavelets Real wavelets are cabable of analysing the degree of smoothness of a signal, detecting breaks in either the signal or its derivatives and the fractal structure of the signal.

Analytic wavelets Analytic wavelets are used to analyze (sound) tones with time dependent frequencies. The use of complex wavelets enables the separation of the phase and amplitude of the signal.

[Example: signal1.mat, Gauss4 wavelet, 1:.2:12, current+all scales, 1-gray, 256 colors]

Definition 7 *(analytic functions)*

(i) *A function $f \in L^2(\mathbb{R})$ is called analytic if $\widehat{f}(\omega) = 0$ for $\omega < 0$.*

(ii) *for a given $f \in L^2(\mathbb{R})$, the analytic part f_{a} of the function f is defined as*

$$f_{\mathsf{a}} = 2\mathcal{F}^{-1}\left(\widehat{f}\chi_{[0,\infty)}\right).$$

In words

$$\widehat{f}_{\mathsf{a}} = \begin{cases} 2\widehat{f}(\omega), & \omega \geq 0 \\ 0, & \omega < 0, \end{cases}$$

which means that the analytic part of a function is the inverse Fourier transform of twice the Fourier transform of the original function reduced to zero for negative values of ω.

An anlytic function f is completely determined by its real part. To see this we write

$$f(t) = u(t) + iv(t),$$

then

$$u(t) = \frac{f(t) + \overline{f(t)}}{2}$$

and

$$\begin{aligned} \widehat{u}(\omega) &= \frac{\widehat{f}(\omega) + \widehat{\overline{f}}(\omega)}{2} \\ &= \frac{\widehat{f}(\omega) + \overline{\widehat{f}}(-\omega)}{2}. \end{aligned}$$

If $\omega \geq 0$ then $\widehat{u}(\omega) = \frac{\widehat{f}(\omega)}{2}$. Hence,

$$\widehat{f}(\omega) = \begin{cases} 2\widehat{u}(\omega), & \omega \geq 0 \\ 0, & \omega < 0. \end{cases}$$

Exercise 3 Show that if $f \in L^2(\mathbb{R})$ is real, then $f = \operatorname{Re}(f_{\mathsf{a}})$. [Hint: if f is real then $\overline{\widehat{f}}(\omega) = \widehat{f}(-\omega)$.]

Theorem 8 *(properties of analytic wavelet transforms)*
Suppose ψ is an analytic wavelet. Then

(a) *for any $f \in L^2(\mathbb{R})$*

$$W_\psi f(a,b) = \frac{1}{2} W_\psi f_{\mathsf{a}}(a,b).$$

(b) *If $f \in L^2(\mathbb{R})$ is real then*

$$f(t) = \frac{1}{2} \operatorname{Re}\left[\int_0^\infty \int_{-\infty}^\infty W_\psi f_{\mathsf{a}}(a,b) D_a \psi(t-d) w(a) \, db da\right]$$

and

$$\|f\|^2 = \frac{1}{2} \|W_\psi f_{\mathsf{a}}\|^2.$$

Proof. We prove only part (a), leaving part (b) as an exercise. Since

$$W_\psi f(a,b) = f * D_a \mathcal{R}\overline{\psi}(b),$$

we may take the Fourier transform on both sides with respect to b and get

$$\begin{aligned}\widehat{W_\psi f(a,\cdot)}(\omega) &= \widehat{f}(\omega)\, D_{1/a}\overline{\widehat{\psi}}(\omega) \\ &= \frac{1}{2}\widehat{f_{\mathsf{a}}}(\omega)\, D_{1/a}\overline{\widehat{\psi}}(\omega),\end{aligned}$$

since $\widehat{\psi}(\omega) = 0$ for $\omega < 0$. Thus,

$$\begin{aligned}W_\psi f(a,b) &= \frac{1}{2} f_{\mathsf{a}} * D_a \mathcal{R}\overline{\psi}(b) \\ &= \frac{1}{2} W_\psi f_{\mathsf{a}}(a,b).\end{aligned}$$

■

Exercise 4 Prove part (b) of the previous theorem.

An analytic wavelet can be constructed by taking an even function $\widehat{g} \in L^2(\mathbb{R})$ with unit norm and support contained in the interval $(-\eta, \eta)$ and define the wavelet ψ by

$$\psi(t) = g(t)\, e^{2\pi i \eta t}. \tag{2}$$

Then ψ satisfies all the conditions of a wavelet, $\widehat{\psi}(\omega) = 0$ for $\omega < 0$ (i.e., ψ is an analytic wavelet) and its energy center $(t_c, \omega_c) = (0, \eta)$.

Exercise 5 Check the above properties of the analytic wavelet ψ given by (2).

Exercise 6 Construct an analytic wavelet from the function g for which $\widehat{g}(\omega) = \chi_{[-1/2,1/2]}(\omega)$.

Approximately analytic wavelets If the function $\widehat{g}$ is such that $\widehat{g}(\omega) \approx 0$ for $|\omega| > \eta$ then ψ is considered "approximately analytic". An example is the *Gabor wavelet* for which

$$\begin{aligned}\widehat{g}(\omega) &= \sqrt{2\pi}\sigma e^{-2\pi\sigma^2\omega^2}, \\ g(t) &= \frac{1}{(\sigma^2\pi)^{1/4}} e^{-\frac{t^2}{2\sigma^2}}.\end{aligned}$$

If $\sigma^2\eta^2 \gg 1$ then $\widehat{g}(\omega) \approx 0$ for $|\omega| > \eta$.

Exercise 7 For the Gabor wavelet, compute the energy center and spreads and check that $\sigma_t \sigma_\omega \geq \frac{1}{2}$. Make plots of $\widehat{g}(\omega)$ for various values of σ and experiment with the choices of η such that $\widehat{g}(\omega) \approx 0$ for $|\omega| > \eta$.